H. Saito
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ON A DECOMPOSITION OF SPACES OF CUSP FORMS AND TRACE FORMULA OF HECKE OPERATORS

HIROSHI SAITO

Introduction

For a positive integer N, put

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \,|\, c \equiv 0 \pmod{N} \right\}.$$

For a positive integer κ and a Dirichlet character ψ modulo N, let $S_\kappa(N, \psi)$ denote the space of holomorphic cusp forms for $\Gamma_0(N)$ of weight κ and character ψ. For a positive integer n prime to N, the Hecke operator T_n is defined on $S_\kappa(N, \psi)$, and in the case where $\kappa \geq 2$, an explicit formula for the trace $\mathrm{tr}\, T_n$ of T_n is known by Eichler [6] and Hijikata [8]. But for higher levels, in particular, when N contains a power of a prime as a factor, this formula is not suitable for numerical computations. It is natural to ask a decomposition of $S_\kappa(N, \psi)$ stable under the action of Hecke operators and a formula for $\mathrm{tr}\, T_n$ on each subspace. In fact, when ψ is the trivial character ψ_1, Yamauchi [18] gave a decomposition of $S_\kappa(N, \psi_1)$ and a formula for $\mathrm{tr}\, T_n$ on each subspace by means of the normalizers of $\Gamma_0(N)$. In the case where $N = p^\nu$ with a prime p, $S_\kappa(p^\nu, \psi_1)$ is divided into two subspaces by this decomposition. When $\nu \geq 2$, in Saito-Yamauchi [11] another decomposition of $S_\kappa(p^\nu, \psi_1)$ into four subspaces and the formulas for $\mathrm{tr}\, T_n$ on these subspaces were given by using the normalizer $W = \begin{pmatrix} 0 & -1 \\ p^\nu & 0 \end{pmatrix}$ of $\Gamma_0(p^\nu)$ and the twisting operator R_ε for ε the quadratic residue symbol modulo p. In this paper, we shall generalize these results. In § 1, we define an operator U_χ on $S_\kappa(N, \psi)$ for a character χ which satisfies a certain condition. This operator is a generalization of $R_\varepsilon W R_\varepsilon W$ in [11]. In a similar way as in [11], we can give a formula for $\mathrm{tr}\, U_\chi T_n$ and also for $\mathrm{tr}\, U_\chi W T_n$ with a normalizer W of $\Gamma_0(N)$ when ψ is trivial (§ 2. Th. 2.5. and Th. 2.9.). In § 3, we shall prove a multiplicative property of U_χ. This

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