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ON THE GROUP OF AUTOMORPHISMS OF A HOPF MAP

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§1. Introduction

Let K be an infinite field of characteristic not 2. Let q_X, q_Y be nonsingular quadratic forms on vector spaces X, Y over K, respectively. Assume that there is a bilinear map $B: X \times Y \to Y$ such that $q_Y(B(x, y)) = q_X(x)q_Y(y)$. To each such triple $\{q_X, q_Y, B\}$ one associates the Hopf map $h: Z = X \times Y \to W = K \times Y$ by $h(z) = (q_X(x) - q_Y(y), 2B(x, y))$, $z = (x, y)$. Denote by q_Z, q_W quadratic forms on Z, W, respectively, defined by $q_Z(z) = q_X(x) + q_Y(y)$, $q_W(w) = u^2 + q_Y(v)$, $w = (u, v)$. One sees easily that $q_W(h(z)) = q_Z(z)^2$, which means that h sends a sphere into a sphere. We shall denote by G the group of automorphisms of h, i.e. the group formed by all automorphisms $s \in GL(Z)$ such that $h(sz) = h(z)$ for all $z \in Z$. After the model of the relationship of quadratic forms and orthogonal groups, it is natural to ask questions such as: what is the structure of G, how G acts on the fibre, what the 1st cohomology of G looks like, and how about the Hasse principle for G when the ground field is a number field? In the present paper, we shall limit our considerations to the case where X is an algebra with 1 over K together with a nonsingular quadratic form q_X such that $q_X(xy) = q_X(x)q_X(y)$, $x, y \in X$. Thanks to a theorem due to A. Hurwitz, such algebras, called composition algebras, are completely determined (cf. [1], Theorem 3.25, p. 73). Namely, an algebra (X, q_X) is one of the following: (I) $X = K$; (II) $X = K + K$; (III) $X =$ a quadratic extension of K; (IV) X = a quaternion algebra over K; (V) $X =$ a Cayley algebra over K. Furthermore, if $X = K$, then $q_X(x) = x^2$; otherwise q_X is the norm form on X. Except for some easy arguments which work for an arbitrary triple $\{q_X, q_Y, B\}$, our results depend on the above theorem of Hurwitz. One can answer completely the questions mentioned above. For the general case, I have, at present, no definite idea how to handle it except the feeling that one needs detailed study of representations of Clifford algebras.

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