S. Shirai
Nagoya Math. J.
Vol. 75 (1979), 121-131

ON GALOIS GROUPS OF CLASS TWO EXTENSIONS OVER THE RATIONAL NUMBER FIELD

SUSUMU SHIRAI

Introduction

Let $\boldsymbol{Q}$ be the rational number field, $K/\boldsymbol{Q}$ be a maximal[1] Abelian extension whose degree is some power of a prime ℓ, and let $f(K)$ be the conductor of $K/\boldsymbol{Q}$; if $\ell=2$, let K be complex, and if in addition $f(K)\equiv 0 \pmod 2$, let $f(K)\equiv 0 \pmod{16}$. Denote by $\mathfrak{F}(K)$ the Geschlechtermodul of K over $\boldsymbol{Q}$ and by $\hat{K}$ the maximal central ℓ-extension of $K/\boldsymbol{Q}$ contained in the ray class field mod $\mathfrak{F}(K)$ of K. A. Fröhlich [1, Theorem 4] completely determined the Galois group of $\hat{K}$ over $\boldsymbol{Q}$ in purely rational terms. The proof is based on [1, Theorem 3], though he did not write the proof in the case $f(K)\equiv 0 \pmod{16}$. Moreover he gave a classification theory of all class two extensions over $\boldsymbol{Q}$ whose degree is a power of ℓ. Hence we know the set of fields of nilpotency class two over $\boldsymbol{Q}$, because a finite nilpotent group is a direct product of all its Sylow subgroups. But the theory becomes cumbersome, and it is desirable to reconstruct a more elementary one.

In the present paper we take the m-th cyclotomic field K_m as K and the central class field $\hat{K}_{mp_\infty}$[2] mod mp_∞ of $K_m/\boldsymbol{Q}$ as $\hat{K}$, where p_∞ stands for the real prime divisor of $\boldsymbol{Q}$. Then we determine the Galois group of $\hat{K}_{mp_\infty}$ over $\boldsymbol{Q}$ by refining the methods used in [1] when $(m, 16)\neq 8$ (Theorem 6). The proof is based on [5, Theorem 32] which is a generalization of [1, Theorem 3] to a cyclotomic field over $\boldsymbol{Q}$. We have already shown in [5] that if $L/\boldsymbol{Q}$ is a normal extension whose Galois group is of nilpotency class two, then there exists a positive integer m such that $L\subset\hat{K}_{mp_\infty}$. Thus as regards Galois groups, we possess the set of all nilpotency class

Received July 19, 1978.

1) This is of sense of Fröhlich [1], which implies that the union of all Abelian ℓ-extensions defined mod $f(K)$ over $\boldsymbol{Q}$ is K itself, in other words, the ℓ-genus field of $K/\boldsymbol{Q}$ contained in the ray class field mod $\mathfrak{F}(K)$ of K coincides with K.

2) See [5, §3].