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ON ALGEBRAIC GROUPS DEFINED BY JORDAN PAIRS

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Introduction

Let G be an algebraic group over a field k, and let ψ be an action of the multiplicative group k_m of k on G by automorphisms. We say ψ is an *elementary action* if it has only the weights 0, ± 1; more precisely, if there exist subgroups H, U^+, U^- of G such that (i) H is fixed under ψ, (ii) U^+ and U^- are vector groups and $\psi_t(x) = t^{\pm 1}x$ for $t \in k_m$, $x \in U^{\pm}$, (iii) $\Omega = U^- \cdot H \cdot U^+$ is open in G, and (iv) G is generated by H, U^+, U^-. This situation is characteristic for the complexifications of the automorphism groups of bounded symmetric domains (see, e.g., [9, 16]). A typical example is $G = \mathbf{GL}_n$ with (matrices being decomposed into 4 blocks) ψ given by

$$\psi_t\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & tb \\ t^{-1}c & d \end{pmatrix} .$$

If G is reductive and U^+ and U^- are one-dimensional, then an elementary action is essentially equivalent to an elementary system in the sense of Demazure [7, Exp. XX], with the technical difference that we consider an external torus action instead of a sub-torus of G acting by conjugation.

After some preliminaries, our first goal (§ 4) is to find relations describing the structure of G in terms of the generators H, U^+, U^-. Since H normalizes U^+ and U^- this essentially amounts to a formula expressing products in $U^+ \cdot U^-$ in terms of their components in $\Omega = U^- \cdot H \cdot U^+$. In more detail, let $\mathfrak{B}^{\pm}$ be the Lie algebra of $U^{\pm}$. Then there are ψ-equivariant isomorphism $\exp\colon \mathfrak{B}^{\pm} \to U^{\pm}$, and there is a unique Jordan pair structure on $\mathfrak{B} = (\mathfrak{B}^+, \mathfrak{B}^-)$ such that, for $x \in \mathfrak{B}^+$, $y \in \mathfrak{B}^-$, the product $\exp(x) \cdot \exp(y)$ belongs to Ω if and only if (x, y) is quasi-invertible, and in this case

$$(*) \qquad \exp(x) \cdot \exp(y) = \exp(y^x) \cdot b(x, y) \cdot \exp(x^y) .$$

Here x^y, y^x denotes the quasi-inverse in $\mathfrak{B}$ and b is a morphism from the set of quasi-invertible pairs of $\mathfrak{B}$ into H which has properties analogous to the "Bergmann transformations" $B(x, y)$ of a Jordan pair. The formula