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ON THE SPECTRUM OF PERIODIC ELLIPTIC OPERATORS

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§ 0. Introduction

It was observed in [Su5] that the spectrum of a *periodic* Schrödinger operator on a Riemannian manifold has *band structure* if the transformation group acting on the manifold satisfies the *Kadison property* (see below for the definition). Here band structure means that the spectrum is a union of mutually disjoint, possibly degenerate closed intervals, such that any compact subset of $\mathbf{R}$ meets only finitely many. The purpose of this paper is to show, by a slightly different method, that this is also true for general periodic elliptic self-adjoint operators.

Let X be a Riemannian manifold of dimension n on which a discrete group Γ acts isometrically, effectively, and properly discontinuously. We assume that the quotient space $\Gamma \backslash X$ (which may have singularities) is compact. Let E be a Γ-equivariant hermitian vector bundle over X, and $D : C^\infty(E) \longrightarrow C^\infty(E)$ a formally self-adjoint elliptic operator which commutes with the Γ-action. For short, we call such a D a Γ-*periodic* operator. It is easy to show (see Section 1) that the symmetric operator D with the domain $C_0^\infty(E)$ is *essentially self-adjoint*, so that D has a unique self-adjoint extension in the Hilbert space $L^2(E)$ of square integrable section of E, which we denote also by D by a slight abuse of notation.

Let $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$ denote the tensor product of the *reduced group* C^*-*algebra* of Γ with the algebra $\mathcal{K}$ of compact operators on a separable Hilbert space of infinite dimension, and by tr_Γ the canonical trace on $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$. We define the *Kadison constant* $C(\Gamma)$ by

$$C(\Gamma) = \inf \{\mathrm{tr}_\Gamma P \,;\, P \text{ is a non-zero projection in } C^*_{\mathrm{red}}(\Gamma, \mathcal{K})\}.$$

By definition, Γ is said to satisfy the Kadison property if $C(\Gamma) > 0$. It is a conjecture proposed by Kadison that, if Γ is torsion free, then $C(\Gamma) = 1$. A

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