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THE FUNDAMENTAL UNIT AND BOUNDS FOR CLASS NUMBERS OF REAL QUADRATIC FIELDS

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Introduction

Although class number one problem for imaginary quadratic fields was solved in 1966 by A. Baker [3] and by H. M. Stark [25] independently, the problem for real quadratic fields remains still unsettled. However, since papers by Ankeny-Chowla-Hasse [2] and H. Hasse [9], many papers concerning this problem or giving estimate for class numbers of real quadratic fields from below have appeared. There are three methods used there, namely the first is related with quadratic diophantine equations ([2], [9], [27, 28, 29, 31], [17]), and the second is related with continued fraction expantions ([8], [4], [16], [14], [18]). The third is related with Dirichlet's classical class number formula

$$h_D = (2\log \varepsilon_D)^{-1}\sqrt{D}\,L(1, \chi_D)\,,$$

where $L(1, \chi_D)$ is the value at $s = 1$ of the L-function

$$L(s, \chi_D) = \sum_{n=1}^{\infty} \chi_D(n) n^{-s}$$

with Kronecker character χ_D belonging to the real quadraric field $\mathbf{Q}(\sqrt{D})$ ([12], [30], [20]). There, T. Tatuzawa's lower bound for $L(1, \chi_D)$:

$$L(1, \chi_D) > 0.655\,(c/D^c) \quad \text{(with one possible exception of } D)$$

plays very important role (cf. [26], [10]).

On the other hand, regarding estimate for the class number of real quadratic fields from above, there are two methods. One of them uses L. K. Hua's upper bound (cf. [11]) for $L(1, \chi_D)$:

$$L(1, \chi_D) < 2^{-1}\log D + 1$$

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