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Nagoya Math. J.
Vol. 123 (1991), 39–76

NUMBER OF GENERATORS OF IDEALS

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Dedicated to Professor Hideyuki Matsumura on his sixtieth birthday

Introduction

Let I be a homogeneous ideal of a polynomial ring over a field, $\nu(I)$ the number of elements of any minimal basis of I, $e = e(I)$ the multiplicity or degree of R/I, $h = h(I)$ the height or codimension of I, $i = \operatorname{indeg}(I)$ the initial degree of I, i.e. the minimal degree of non zero elements of I.

This paper is mainly devoted to find bounds for $\nu(I)$ when I ranges over large classes of ideals. For instance we get bounds when I ranges over the set of perfect ideals with preassigned codimension and multiplicity and when I ranges over the set of perfect ideals with preassigned codimension, multiplicity and initial degree. Moreover all the bounds are sharp since they are attained by suitable ideals. Now let us make some historical remarks.

It is a classical result of Krull that $h(I) \leq \nu(I)$, and Macaulay showed that there is no upper bound for $\nu(I)$ when I ranges over the set of perfect codimension 2 prime ideals of $k[x, y, z]$. But what happens if $e(I)$ is given? Many authors studied a more general problem, allowing the ambient ring R to be a Cohen-Macaulay ring. If R is specialized to be a polynomial ring or a regular local ring, we deduce from their results the following bounds:

$\nu \leq e^{h-1} + h - 1$ — Sally (1976)

$\nu \leq (h!/\sqrt[h]{h!})e^{1-1/h} + h - 1$ — Boratynski-Eisenbud-Rees (1979)

$\nu \leq 1 + [(h-1)^2/h]e + (h^2-1)/h - \binom{h}{2}$ — Valla (1981)

Received April 4, 1990.

1) The first author was partly supported by the Department of Mathematics of the University of Genova.

2) The second author was partly supported by the U.S. Army Research Office through the Mathematical Sciences Institute of Cornell University.