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THE GENERALIZED DIVISOR PROBLEM AND THE RIEMANN HYPOTHESIS

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Introduction

Let $d_z(n)$ be a multiplicative function defined by

$$\zeta^z(s) = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^s} \quad (\sigma > 1)$$

where $s = \sigma + it$, z is a complex number, and $\zeta(s)$ is the Riemann zeta function. Here $\zeta^z(s) = \exp(z \log \zeta(s))$ and let $\log \zeta(s)$ take real values for real $s > 1$. We note that if z is a natural number $d_z(n)$ coincides with the divisor function appearing in the Dirichlet-Piltz divisor problem, and $d_{-1}(n)$ with the Möbious function.

The generalized divisor problem is concerned with finding an asymptotic formula for $\sum_{n \leq x} d_z(n)$, which was observed for real $z > 0$ by A. Kienast [6] and K. Iseki [4] independently. A. Selberg [8] considered for all complex z, his result being

$$D_z(x) \equiv \sum_{n \leq x} d_z(n) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O(x(\log x)^{\Re z-2}) \tag{1}$$

uniformly for $|z| \leq A$, $x \geq 2$, where A is any fixed positive number.

Next, let $\pi_k(x)$ be the number of integers $\leq x$ which are products of k distinct primes. For $k = 1$, $\pi_k(x)$ reduces to $\pi(x)$, the number of primes not exceeding x. C. F. Gauss stated empirically that $\pi_2(x) \sim x(\log \log x)/\log x$, and, by using the prime number theorem, E. Landau proved that $\pi_k(x) \sim x(\log \log x)^{k-1}/(k-1)! \log x$. Selberg considered $D_z(x)$ not only for its own sake but also with an intension to derive

$$\pi_k(x) = \frac{xQ(\log \log x)}{\log x} + O\left(\frac{x(\log \log x)^k}{k!(\log x)^2}\right) \tag{2}$$

uniformly for $1 \leq k \leq A \log \log x$, where $Q(x)$ is oplynomial of degree