T. Murai
Nagoya Math. J.
Vol. 122 (1991), 19-42

ANALYTIC CAPACITY FOR TWO SEGMENTS

TAKAFUMI MURAI

§ 1. Introduction

The analytic capacity $\gamma(E)$ of a compact set E in the complex plane $\mathbf{C}$ is defined by $\gamma(E) = \sup|f'(\infty)|$, where $-f'(\infty)$ is the $1/z$-coefficient of $f(\zeta)$ at infinity and the supremum is taken over all bounded analytic functions $f(\zeta)$ outside E with supremum norm less than or equal to 1. Analytic capacity $\gamma(\cdot)$ plays various important roles in the theory of bounded analytic functions.

It is known that $\gamma(E) \leq |E|$, where $|\cdot|$ is the (generalized) length (i.e., the 1-dimension Hausdorff measure [3, CHAP. III]) and that the inverse relation does not exist, in general. In fact, Vitushkin [14] constructs an example of a set with positive length but zero analytic capacity, and Garnett [3, p. 87] also points out that the planar Cantor set with ratio 1/4

$$E(1/4) = \bigcap_{n=0}^{\infty} E_n$$

satisfies the same property. Here E_0 is the unit square $[0, 1] \times [0, 1]$ and E_n is inductively defined from E_{n-1} with each square Q of E_{n-1} replaced by four squares with sides 4^{-n} in the four corners of Q. The set E_n is a union of 4^n squares with sides 4^{-n}, and the projections of these 4^n squares to the line $\mathscr{L}$: $y = x/2$ do not mutually overlap. Hence if we choose $\mathscr{L}$ as a new axis, then E_n seems like a discontinuous graph. From this point of view, the author [8, CHAP. III] defined cranks and studied their analytic capacities: Cranks are nothing but deformations of sets of Vitushkin-Garnett type, however, these discontinuous graphs simplify the computation of analytic capacity and enable us to construct various examples [8, Theorem F], [9]. Hence clarifying the geometric meaning of cranks is important and would be applicable to study analytic capacities of general sets. (Cranks are closely related to fractals (Mandelbrot [6]).)

Received April 21, 1989.