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Nagoya Math. J.
Vol. 121 (1991), 195–199

ON THE STRONG UNIMODALITY OF LÉVY PROCESSES

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§ 1. Introduction and results

A measure $\mu(dx)$ on R is said to be unimodal with mode a if $\mu(dx) = c\delta_a(dx) + f(x)dx$, where $c \geqq 0$, $\delta_a(dx)$ is the delta measure at a and $f(x)$ is non-decreasing for $x < a$ and non-increasing for $x > a$. A measure $\mu(dx) = \sum_{n=-\infty}^{\infty} p_n\delta_n(dx)$ on $Z = \{0, \pm 1, \pm 2, \cdots\}$ is said to be unimodal with mode a if p_n is non-decreasing for $n \leqq a$ and non-increasing for $n \geqq a$. A probability measure $\mu(dx)$ on R (resp. on Z) is said to be strongly unimodal on R (resp. on Z) if, for every unimodal probability measure $\eta(dx)$ on R (resp. on Z), the convolution $\mu * \eta(dx)$ is unimodal on R (resp. on Z). Let X_t, $t \in [0, \infty)$, be a Lévy process (that is, a process with stationary independent increments starting at the origin) on R (resp. on Z) with the Lévy measure $\nu(dx)$. The process X_t is said to be unimodal on R (resp. on Z) if, for every $t > 0$, the distribution of X_t is unimodal on R (resp. on Z). It is said to be strongly unimodal on R (resp. on Z) if, for every $t > 0$, the distribution of X_t is strongly unimodal on R (resp. on Z). In this paper we shall characterize strongly unimodal Lévy processes on R and Z.

THEOREM 1. *Let X_t be a Lévy process on R. Then X_t is strongly unimodal on R if and only if*

$$X_t = \sigma B(t) + \gamma t,$$

where $B(t)$ is a Brownian motion and σ and γ are constants, $\sigma \geqq 0$.

THEOREM 2. *Let X_t be a Lévy process on Z. Then X_t is strongly unimodal on Z if and only if*

$$X_t = X_{at}^{(1)} - X_{bt}^{(2)},$$

Received June 25, 1990.