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EXPLICIT DESCRIPTIONS OF TRACE RINGS OF GENERIC 2 BY 2 MATRICES

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§1. Introduction

Let K be a field of characteristic zero and let

$$X_1 = (x_{ij}(1)), \cdots, X_m = (x_{ij}(m)), \qquad m \geq 2,$$

be m generic n by n matrices over K. That is, $x_{ij}(k)$ are independent commuting indeterminates over K. The K-subalgebra generated by $X_1, \cdots, X_m$ is called a ring of n by n generic matrices and is denoted by $R(n, m)$. Let $M_n(K[x_{ij}(k)])$ denote the n by n matrix algebra over the polynomial ring $K[x_{ij}(k)]$. The ring $R(n, m)$ is a K-subalgebra of $M_n(K[x_{ij}(k)])$. Let $C(n, m)$ be the subring of the polynomial ring $K[x_{ij}(k)]$ generated by all traces $\mathrm{Tr}(X_{i_1} \cdots X_{i_\alpha})$, where $X_{i_1} \cdots X_{i_\alpha}$ is a monomial in the generic matrices $X_1, \cdots, X_m$. The trace ring $T(n, m)$ of m generic n by n matrices is the K-subalgebra of $M_n(K[x_{ij}(k)])$ generated by $R(n, m)$ and $C(n, m)$. Here we identify elements of $C(n, m)$ with scalar matrices.

In this paper we will be concerned with the trace ring $T(2, m)$ of generic 2 by 2 matrices. L. Le Bruyn [1. Chap. 3, Theorem 5.1] proved that $T(2, m)$ is a Cohen-Macaulay module over $C(n, m)$. Apart from this general result, very little is known about explicit structure on $T(2, m)$. Explicit descriptions of $T(2, m)$ are known only for $m \leq 4$ (cf. [2], [3], [4]) and except these cases nothing is known on an explicit description of $T(2, m)$. In this paper we will give explicit descriptions of $T(2, m)$ for all m.

A Young tableau on numbers $1, 2, \cdots, m$

$$Y = \begin{bmatrix} i_1 & i_2 \cdots i_r \\ j_1 & j_2 \cdots j_r \end{bmatrix}$$

is called standard if the entries strictly increase down columns and non-decrease across rows. Let $X_1, \cdots, X_m$ be m generic 2 by 2 matrices. We

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