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ON THE CLASS NUMBER AND UNIT INDEX OF SIMPLEST QUARTIC FIELDS

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§ 1. Introduction

The term "simplest" field has been used to describe certain totally real, cyclic number fields of degrees 2, 3, 4, 5, 6, and 8. For each of these degrees, the fields are defined by a one-parameter family of polynomials with constant term ± 1. The regulator of these "simplest" fields is small in an asymptotic sense: in consequence, the class number of these fields tends to be large.

The simplest quartic fields are defined by adjunction to $\mathbf{Q}$ of a root of

$$(*) \qquad P_t = X^4 - tX^3 - 6X^2 + tX + 1\,, \qquad t \in \mathbf{Z}^+$$

where $t^2 + 16$ is not divisible by an *odd* square [5]. Here t may be specified greater than zero since P_t and P_{-t} generate the same extension. This polynomial is reducible precisely when $t^2 + 16$ is a square, which occurs only for the excluded cases $t = 0, 3$.

Gras [5] shows that the form $T^2 + 16$ represents infinitely many square-free integers, so this family is infinite. As an example of why the odd-square-free restriction is important, note that $t = 22$, for which $t^2 + 16 = 500$, defines the same field as $t = 2$.

Some of the simplest fields arise from torsion elements in $PSL(2, \mathbf{Q})$ acting as linear fractional transformations on one given root [3]. The matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ has order 4; the cyclic Galois action on the roots of $(*)$ is given by $\varepsilon \mapsto (\varepsilon - 1)/(\varepsilon + 1)$.

Remark. References to other examples of simplest fields may be found in [10].

Notation. Throughout K will be a real cyclic quartic field, simplest unless stated otherwise, and k will be the unique quadratic subfield. Let

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