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SUPER CONGRUENCE FOR THE APÉRY NUMBERS

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§0. Introduction

Let, for any $n \geq 0$,

$$a(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \qquad u(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 .$$

R. Apéry's proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ made use of these numbers (see [10]). As a result, many properties of the Apéry numbers were found (see [1]-[9]). In particular, Beukers and Stienstra showed the interesting congruence (see [11, Theorem 13.1]).

THEOREM 1 (Beukers and Stienstra). *Let $p \geq 3$ be a prime, and write*

$$(1) \qquad \sum_{n=1}^{\infty} \lambda_n q^n = q \prod_{n=0}^{\infty} (1 - q^{4n})^6 .$$

Let m, $r \in \mathbf{N}$, m odd, then we have

$$(2) \qquad a\Big(\frac{mp^r - 1}{2}\Big) - \lambda_p a\Big(\frac{mp^{r-1} - 1}{2}\Big) + (-1)^{(p-1)/2} p^2 a\Big(\frac{mp^{r-2} - 1}{2}\Big) \equiv 0 \bmod p^r .$$

Moreover they conjectured that congruence (2) holds $\bmod p^{2r}$ if $p \geq 5$, and they called these congruences *super congruences* in [4] and [11].

In this paper we shall prove the conjecture for $r = 1$.

THEOREM 2. *Let $p \geq 5$ be a prime and $m \in \mathbf{N}$, m odd, then we have*

$$a\Big(\frac{mp - 1}{2}\Big) - \lambda_p a\Big(\frac{m - 1}{2}\Big) \equiv 0 \bmod p^2 .$$

F. Beukers informed me that L. Van Hamme proved the case of $p \equiv 1 \bmod 4$ using properties of the p-adic gamma function (see [7]). We prove the general case involving $p \equiv 3 \bmod 4$ by entirely different method. Our

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