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SECOND PROOF OF THE IRREDUCIBILITY OF THE FIRST DIFFERENTIAL EQUATION OF PAINLEVÉ

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In our paper [U2], we proved the irreducibility of the first differential equation $y'' = 6y^2 + x$ of Painlevé. In that paper we explained the origin of the problem and the importance of giving a rigorous proof. We can say that our method in [U2] is algebraic and finite dimensional in contrast to a prediction of Painlevé who expected a proof depending on the infinite dimensional differential Galois theory. Even nowadays the latter remains to be established. It seems that Painlevé needed an armament with the general theory (the infinite dimensional differential Galois theory) in the controversy with R. Liouville on the mathematical foundation of the proof of the irreducibility of the first differential equation (1902–03). Thus he forgot his earlier idea of proving the irreducibility, which is simple and natural and found in the twenty-first lecture of Leçons de Stockholm given in 1895 (von. 1 [P]): a differential equation $y'' = R(x, y, y')$ (here $R(x, y, y')$ is a rational function of x, y, y' with coefficients in $\mathbb{C}$) free from moving critical points is irreducible if and only if the general solution $y(x_0; y_0, y_0'; x)$ (taking the initial condition y_0, y_0' at x_0) is an essentially transcendental function of (y_0, y_0'). The main result of this paper is a second proof of the irreducibility of $y'' = 6y^2 + x$ based on this idea of Painlevé (§ 3, Theorem (68)). The second proof is analytic as the transcendental correspondence is involved. It looks more indirect than the first proof given in [U2] but it has an advantage. In [U1] we had to make the definition of being irreducible precise or equivalently we had to make the permissble operations clear. Essentially they are the solution of linear differential equations and the substitution in Abelian functions (see § 1). These operations are related with algebraic groups. But in [U2] we proved a better irreducible theorem: impossibility of solving the differential equation $y'' = 6y^2 + x$ by the above 2 operations combined with the solution of first order algebraic differential equations. As we explained above, the first 2 operations are group theoretic but we

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