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ON SEGRE PRODUCTS OF AFFINE SEMIGROUP RINGS

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§ 0. Introduction

Let N denote the set of non-negative integers. An affine semigroup is a finitely generated submonoid S of the additive monoid N^m for some positive integer m. Let $k[S]$ denote the semigroup ring of S over a field k. Then one can identify $k[S]$ with the subring of a polynomial ring $k[t_1, \cdots, t_m]$ generated by the monomials $t^x = t_1^{x_1} \cdots t_m^{x_m}$, $x = (x_1, \cdots, x_m) \in S$.

Let Q denote the field of rational numbers. Let $\sigma: Q^m \to Q$ be a linear functional such that $\sigma(S) \subseteq N$ and $\sigma(x) = 0$, $x \in S$, implies $x = 0$. Then one can define an N-grading on $k[S]$ by setting $\deg t^x = \sigma(x)$ for all $x \in S$. Such a procedure is called specializing to an N-grading [13, p. 190].

If $T \subseteq N^n$ is another affine semigroup and $k[T]$ is specialized to an N-grading by a linear functional $\tau: Q^n \to Q$, then one can define a new affine semigroup $W \subseteq N^m \times N^n$ by setting

$$W := (S \times T) \cap F,$$

where F denotes the set of all elements $(x, y) \in Q^m \times Q^n$ with $\sigma(x) = \tau(y)$. We call $k[W]$ the Segre product of the N-graded rings $k[S]$ and $k[T]$ with respect to σ and τ (cf. [9, p. 125]). The class of rings of the form $k[W]$ includes, for example, the usual Segre product of polynomial rings, the Segre-Veronese graded algebra and the Rees algebras of certain rings generated by monomials. Several authors have been dealt with the Cohen-Macaulayness and the Gorensteiness of Segre products of special classes of affine semigroup rings [1], [2], [3], [4], [16].

The main result of this paper is a combinatorial criterion for $k[W]$ to be a Cohen-Macaulay (res. Gorenstein) in terms of S and T (Theorem 2.1). It is based on a combinatorial criterion of [16] for an affine semigroup ring to be Cohen-Macaulay (res. Gorenstein) which uses certain simplicial complexes associated with the affine semigroup (see Section 1). We shall see that the associated simplicial complexes of W are the joins

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