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DISCRIMINANTS IN THE INVARIANT THEORY OF REFLECTION GROUPS

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§ 1. Introduction

Let V be a complex vector space of dimension l and let $G \subset GL(V)$ be a finite reflection group. Let S be the C-algebra of polynomial functions on V with its usual G-module structure $(gf)(v) = f(g^{-1}v)$. Let R be the subalgebra of G-invariant polynomials. By Chevalley's theorem there exists a set $\mathscr{B} = \{f_1, \cdots, f_l\}$ of homogeneous polynomials such that $R = C[f_1, \cdots, f_l]$. We call $\mathscr{B}$ a set of basic invariants or a *basic set* for G. The degrees $d_i = \deg f_i$ are uniquely determined by G. We agree to number them so that $d_1 \leq \cdots \leq d_l$. The map τ: $V/G \to C^l$ defined by

$$\tau(Gv) = (f_1(v), \cdots, f_l(v)) \tag{1.1}$$

is a bijection. Each reflection in G fixes some hyperplane in V. Let $\mathscr{A} = \mathscr{A}(G)$ be the set of reflecting hyperplanes and let

$$N(G) = \bigcup_{H \in \mathscr{A}} H \tag{1.2}$$

$$M(G) = V - \bigcup_{H \in \mathscr{A}} H . \tag{1.3}$$

If $H \in \mathscr{A}$ let e_H be the order of the (cyclic) subgroup fixing H and let $\alpha_H \in V^*$ be a linear form with kernel H. Since $\prod_{H \in \mathscr{A}} \alpha_H^{e_H} \in R$ we may define a polynomial $\Delta(T_1, \cdots, T_l; \mathscr{B})$ in the indeterminates $T_1, \cdots, T_l$ by

$$\Delta(f_1, \cdots, f_l; \mathscr{B}) = \prod_{H \in \mathscr{A}} \alpha_H^{e_H} . \tag{1.4}$$

We call the polynomial $\Delta(T_1, \cdots, T_l; \mathscr{B})$ the *discriminant* of G relative to $\mathscr{B}$ since it depends on the basic invariants. The hypersurface

$$\tau(N(G)/G) = \{(z_1, \cdots, z_l) \in C^l \mid \Delta(z_1, \cdots, z_l; \mathscr{B}) = 0\} \tag{1.5}$$

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