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THE HESSIAN MAP IN THE INVARIANT THEORY OF REFLECTION GROUPS

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§ 1. Introduction

Let V be a complex vector space of dimension l. Let S be the C-algebra of polynomial functions on V. Let Der_S be the S-module of derivations of S and let $\Omega_S = \mathrm{Hom}_S(\mathrm{Der}_S, S)$ be the dual S-module of differential 1-forms. Let $\{e_i\}$ be a basis for V and let $\{x_i\}$ be the dual basis for V^*. Then $\{D_i = \partial/\partial x_i\}$ and $\{dx_i\}$ are bases for Der_S and Ω_S as S-modules. If $f \in S$, define a map $\mathrm{Hess}(f)\colon \mathrm{Der}_S \to \Omega_S$ by

(1.1) $$\mathrm{Hess}(f)\colon \theta \longrightarrow \sum \theta(D_i f)dx_i \qquad \theta \in \mathrm{Der}_S .$$

Then $\mathrm{Hess}(f)$ is an S-module homomorphism which does not depend on the choice of basis for V. Let $\mathbf{H}(f)$ denote the the matrix of the map $\mathrm{Hess}(f)$ with respect to the pair of bases $\{D_i\}$ and $\{dx_i\}$. Then $\mathbf{H}(f)$ is the usual Hessian matrix of second partial derivatives of f.

Let $G \subset GL(V)$ be a finite unitary reflection group and let $R = S^G$ be the subalgebra of G-invariant polynomials. Both Der_S and Ω_S are G-modules. If $f \in R$ then $\mathrm{Hess}(f)$ induces a homomorphism again denoted $\mathrm{Hess}(f)\colon \mathrm{Der}_S^G \to \Omega_S^G$. If G has a real form, so that G is a Coxeter group, then it has a non-degenerate invariant quadratic form f. Nondegeneracy implies $\det \mathbf{H}(f) \neq 0$. Since $\mathbf{H}(f)$ is a matrix with entries in C it is invertible in $M_l(C)$ so $\mathrm{Hess}(f)\colon \mathrm{Der}_S^G \to \Omega_S^G$ is an isomorphism [15]. The situation is more complicated for unitary reflection groups which do not have a real form. We show that if $\mathrm{Hess}(f)$ is an isomorphism then f is an invariant form of minimal positive degree but this minimality condition is not sufficient. In Theorem (5.10) of this paper we characterize those irreducible unitary reflection groups for which an invariant form f_1 of minimal positive degree induces an isomorphism $\mathrm{Hess}(f_1)\colon \mathrm{Der}_S^G \to \Omega_S^G$.

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