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Nagoya Math. J.
Vol. 106 (1987), 101-111

$\mathfrak{m}$-FULL IDEALS

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Introduction

An ideal $\mathfrak{a}$ of a local ring $(R, \mathfrak{m})$ is called $\mathfrak{m}$-full if $\mathfrak{a}\mathfrak{m}: y = \mathfrak{a}$ for some y in a certain faithfully flat extension of R. The definition is due to Rees (unpublished) and he had obtained some elementary results (also unpublished). The present paper concerns some basic properties of $\mathfrak{m}$-full ideals. One result is the characterization of $\mathfrak{m}$-fullness in terms of the minimal number of generators of ideal, generalizing his result in a low dimensional case (Theorem 2, § 2).

Meanwhile Professor Rees asked me for which ideals is it true that $\mu(\mathfrak{a}) \geqq \mu(\mathfrak{b})$ for all $\mathfrak{b}$ containing $\mathfrak{a}$. Surprisingly enough it turns out that $\mathfrak{m}$-full ideals do have this property (Theorem 3, § 2). To prove this we introduce, in Section 1, three numerical characters Φ, Ψ, $\bar{\mu}$, of $\mathfrak{m}$-primary ideals. Φ and $\bar{\mu}$ are, respectively, the colength and the minimal number of generators of an ideal modulo a general element, and Ψ is the maximum of $\mu(\mathfrak{b})$, where $\mathfrak{b}$ runs over all the ideals containing a given ideal (Definitions 1, 2, 3, § 1).

Theorem 1 in Section 2 shows that these are related by an inequality, from which it immediately follows that an ideal $\mathfrak{a}$ has the property mentioned above if $\mu(\mathfrak{a}) = \Phi(\mathfrak{a}) + \bar{\mu}(\mathfrak{a})$. And Theorem 2 shows that this is precisely equivalent to the $\mathfrak{m}$-fullness of the ideal.

The purpose of Section 3 is to show that the converse of Theorem 3 holds in a 2-dimensional regular local ring, thanks to the equality $\Psi(\mathfrak{a}) = \Phi(\mathfrak{a}) + \bar{\mu}(\mathfrak{a})$ for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$. Also we would like to call attention to the fact that Theorem 1 has grown out of the attempt to generalize Lemma 2 which is easily proved homologically. (See Remark 3, § 3).

In Section 4 we present a theorem of Rees which says that any integrally closed ideal is $\mathfrak{m}$-full, and also we prove an interesting formula

Received October 21, 1985.
Revised April 10, 1986.