B. Singh
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DIFFERENTIAL OPERATORS ON A HYPERSURFACE

BALWANT SINGH*

Introduction

We study differential operators on an affine algebraic variety, especially a hypersurface, in the context of Nakai's Conjecture. We work over a field k of characteristic zero. Let X be a reduced affine algebraic variety over k and let A be its coordinate ring. Let $\mathrm{Diff}_k^n(A)$ be the A-module of differential operators of A over k of order $\leqq n$. Nakai's Conjecture asserts that if $\mathrm{Diff}_k^n(A)$ is generated by $\mathrm{Diff}_k^1(A)$ for every $n \geqq 2$ then A is regular. In 1973 Mount and Villamayor [6] proved this in the case when X is an irreducible curve. In the general case no progress seems to have been made on the conjecture, except for a result of Brown [2], where the assertion is proved under an additional hypothesis.(1) An interesting result proved by Becker [1] and Rego [8] says that Nakai's Conjecture implies the Conjecture of Zariski-Lipman, which is still open in the general case and which asserts that if the module of k-derivations of A is A-projective then A is regular.

Write $A = R/J$, where R is a polynomial ring over k and J is an ideal of R. Let $\mathrm{Diff}_k^n(R, A)$ be the A-module of differential operators of R into A over k of order $\leqq n$. Since R is a polynomial ring, the structure of $\mathrm{Diff}_k^n(R, A)$ is well-known, and $\mathrm{Diff}_k^n(A)$ can be identified with the A-submodule of those $D \in \mathrm{Diff}_k^n(R, A)$ for which $D(J) = 0$. In this paper we first analyze the condition "$D(J) = 0$" in some detail, and prove in Proposition (2.10) that for $D(J)$ to be zero it is sufficient (and necessary) that D and certain other differential operators derived from D vanish on a set of generators of J. This is then used to prove that if X is a hypersurface, i.e. if we can write $A = R/J$ with J principal, then $\mathrm{Diff}_k^2(A)$ is *completely*

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(1) See additional remarks in the last paragraph of this section.