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DIMENSION AND LOWER CENTRAL SUBGROUPS OF METABELIAN P-GROUPS

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To the memory of the late Takehiko Miyata

§ 1. Introduction

It is a well-known result due to Sjogren [9] that if G is a finitely generated p-group then, for all $n \leqq p-1$, the $(n+2)$-th dimension subgroup $D_{n+2}(G)$ of G coincides with $\gamma_{n+2}(G)$, the $(n+2)$-th term of the lower central series of G. This was earlier proved by Moran [5] for $n \leqq p-2$. For $p=2$, Sjogren's result is the best possible as Rips [8] has exhibited a finite 2-group G for which $D_4(G) \neq \gamma_4(G)$ (see also Tahara [10, 11]). In this note we prove that if G is a finitely generated metabelian p-group then, for all $n \leqq p$, $D^2_{n+2}(G) \subseteqq \gamma_{n+2}(G)$. It follows, in particular, that, for p odd, $D_{n+2}(G) = \gamma_{n+2}(G)$ for all $n \leqq p$ and all metabelian p-groups G.

§ 2. Notation and preliminaries

While the central idea of the proof of our main result stems from Gupta [1], with a slight repetition, it is equally convenient to give a self-contained proof using a less cumbersome notation.

Let $\mathfrak{f} = ZF(F-1)$ denote the augmentation ideal of the integral group ring ZF of a free group F freely generated by $x_1, x_2, \cdots, x_m$, $m \geqq 2$. For a fixed prime p, let $(p^{\alpha_1}, p^{\alpha_2}, \cdots, p^{\alpha_m})$, $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_m > 0$ be an m-tuple of p-powers, and let $S = \langle x_1^{p^{\alpha_1}}, x_2^{p^{\alpha_2}}, \cdots, x_m^{p^{\alpha_m}}, F' \rangle$ be the normal subgroup of F so that F/S is abelian. Set $\mathfrak{s} = ZF(S-1)$, the ideal of ZF generated by all elements $s-1$, $s \in S$. For $1 \leqq n \leqq p$, we shall need to investigate the structure of the subgroup $D_{n+2}(\mathfrak{f}\mathfrak{s}) = F \cap (1 + \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+2})$ of F which consists of all elements $w \in F$ such that $w - 1 \in \mathfrak{f}\mathfrak{s} + \mathfrak{f}^{n+2}$. It is clear that $[F', S]\gamma_{n+2}(F) \subseteqq D_{n+2}(\mathfrak{f}\mathfrak{s})$.

Let $w \in D_{n+2}(\mathfrak{f}\mathfrak{s})$ be an arbitrary element. Then $w - 1 \in \mathfrak{f}^2$ and it

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