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THICK SETS AND QUASISYMMETRIC MAPS

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1. Introduction

1.1. **Thickness.** Let E be a real inner product space. For a finite sequence of points $a_0, \ldots, a_k$ in E we let $a_0 \ldots a_k$ denote the convex hull of the set $\{a_0, \ldots, a_k\}$. If these points are affinely independent, the set $\Delta = a_0 \ldots a_k$ is a k-simplex with vertices $a_0, \ldots, a_k$. It has a well-defined k-volume written as $m_k(\Delta)$ or briefly as $m(\Delta)$. We are interested in sets $A \subset E$ which are "nowhere too flat in dimension k". More precisely, suppose that $A \subset E$, $q > 0$ and that k is a positive integer. We let $\bar{B}(x, r)$ denote the closed ball with center x and radius r. We say that A is *(q, k)-thick* if for each $x \in A$ and $r > 0$ such that $A \setminus \bar{B}(x, r) \neq \emptyset$ there is a k-simplex Δ with vertices in $A \cap \bar{B}(x, r)$ such that $m_k(\Delta) \geq qr^k$.

It is easy to see that the closure $\bar{A}$ of a (q, k)-thick set A is (q', k)-thick for each $q' < q$. In the case $\dim E < \infty$, $\bar{A}$ is in fact (q, k)-thick. Conversely, if $\bar{A}$ is (q, k)-thick, A is (q', k)-thick for all $q' < q$. Without essential loss of generality, it is thus sufficient to consider only closed sets $A \subset E$.

We also say that A is *k-thick* if A is (q, k)-thick for some $q > 0$. It is easy to see that a p-thick set is k-thick for all $k \leq p$.

1.2. Examples. We consider sets in the Euclidean n-space R^n. A set $A \subset R^n$ can be k-thick only for $k \leq n$. A k-dimensional ball and a k-cube are clearly k-thick but not p-thick for $p > k$. The Cantor middle-third set is 1-thick. If A is an arc which has a tangent at some point, A is not 2-thick. In particular, rectifiable arcs are not 2-thick. On the other hand, the Koch snowflake curve in R^2 is 2-thick. A c-John domain [NV], 2.26, and its closure in R^n are (q, n)-thick with $q = q(c, n)$.

1.3. **Background.** Thick sets arise naturally from various questions of analysis. For example, in [Vä3], Th. 6.2, it was proved that if A is compact and

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