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ON A CLASS OF NUMBERS GENERATED BY DIFFERENCIAL EQUATIONS RELATED WITH ALGEBRAIC GROUPS

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Introduction

In this paper we propose a new category $\mathbf{Q}^{cl}$ of complex numbers which contains π, e and the set of algebraic numbers. In fact this category contains most of the numbers studied so far in number theory. An element of the category is here called a classical number. The category of the classical numbers forms an algebraically closed field and consists of countably many numbers. The definition depends on algebraic differential equations related with algebraic groups. Throughout the paper unless otherwise stated, we deal with functions of one variable and a differential equation is an ordinary differential equation. We are inspired of the Leçons de Stockholm of Painlevé [P]. His objective was to discover new transcendental functions defined by algebraic differential equations generalizing the Weierstrass $\wp$-function. To this end there are two major tasks to be done. The first is to find candidates of algebraic differential equations which may define new functions. We are concerned with the second which is to check whether the candidates really define new functions. So he introduced a class of functions inductively defined from the field $\mathbf{C}(x)$ of the rational functions by admissible operations. In [U4], [U6] we analyzed his operations and introduced the permissible operations (O), (P1), (P2),...,(P5) (cf. §1). We proved that allowing the permissible operations is equivalent to admitting G-primitive extensions of differential fields in the language of Kolchin. We defined the field of the classical functions as a field of the meromorphic functions obtained from the field $\mathbf{C}(x)$ of the rational functions by a finite iteration of permissible operations (cf. [U4]). All the functions related with differential equations in [WW] are classical in this sense, for example $\exp x$, $\log x$, the hypergeometric functions, the elliptic functions and so on. The irreducibility theorem says that we can not solve the first differential equation of Painlevé starting from the field $\mathbf{C}(X)$ of the rational functions by any finite iteration of the

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