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OSCILLATION OF MODES OF SOME SEMI-STABLE LÉVY PROCESSES

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§1. Introduction

In this paper it is shown that there is a unimodal Lévy process with oscillating mode. After the author first constructed an example of such a self-decomposable process, Sato pointed out that it belongs to the class of semi-stable processes with $\beta < 0$. We prove that all non-symmetric semi-stable self-decomposable processes with $\beta < 0$ have oscillating modes.

A measure μ on $\mathbf{R}$ is said to be *unimodal* with mode $a \in \mathbf{R}$ if $\mu(dx) = c\,\delta_a(dx) + f(x)dx$, where c is non-negative, δ_a is the delta measure at a and $f(x)$ is non-decreasing on $(-\infty, a)$ and non-increasing on (a, ∞). If a measure μ is unimodal, then either its mode is unique or the set of its modes is a closed interval. Let $\{X_t\}$, $t \in [0, \infty)$, be a Lévy process on $\mathbf{R}$ (that is, a stochastically continuous process with stationary independent increments starting at the origin) and let μ_t be the distribution of X_t. The Lévy process $\{X_t\}$ is said to be unimodal if μ_t is unimodal for each t. When a Lévy process $\{X_t\}$ is unimodal, we denote a mode of μ_t by $a(t)$. In case the set of modes of μ_t is a closed interval, there is freedom of choice of $a(t)$. The Lévy process $\{X_t\}$ is said to be *self-decomposable* if μ_t is an L distribution for each t. A self-decomposable Lévy process is simply called a self-decomposable process. Yamazato proves in the celebrated paper [16] that every self-decomposable process is unimodal. We say that a Lévy process $\{X_t\}$ is semi-stable if there exist real numbers β and γ such that $0 < |\beta| < 1$, $1 < \gamma$, $\gamma = |\beta|^{-\lambda}$ $(0 < \lambda \leqq 2)$ and

$$\hat{\mu}_t(z) = \hat{\mu}_{\gamma t}(\beta z) \tag{1.1}$$

for every $z \in \mathbf{R}$ and every $t \geqq 0$, where

$$\hat{\mu}_t(z) = \int_0^{\infty} e^{izx}\,\mu_t(dx). \tag{1.2}$$

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