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A NOTE ON THE FØLNER CONDITION FOR AMENABILITY

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Dedicated to Professor Nobunori Ikebe for his 60th birthday

1. Introduction

Let G be a countably generated discrete group. A right-invariant *mean* μ on G is a bounded linear functional of the space $L^{\infty}(G)$ of bounded functions on G having the property:

(1) $\inf_{x\in G} \varphi(x) \leq \mu(\varphi) \leq \sup_{x\in G} \varphi(x)$ for $\varphi \in L^{\infty}(G)$,

(2) for every $g \in G$, $\mu(g\cdot\varphi) = \mu(\varphi)$, where $g\cdot\varphi(x) = \varphi(xg)$.

We say that G is *amenable* if it is equipped with a right-invariant mean. Finite groups, abelian groups, in fact, groups of subexponential growth are amenable. Solvable group are also amenable. Subgroups and quotients of amenable groups are amenable. On the other hand, free groups having two generators and over are non-amenable.

In his paper [4], Følner gives the following combinatorial characterization.

THEOREM (Følner). *A countably generated discrete group G is amenable if and only if, for every positive ε and arbitrary finite subset A of G, there exists a non-empty finite subset E of G such that*

$$\#(E\cdot a \setminus E) \leq \varepsilon \# E \quad \text{for every } a \in A,$$

where $\# F$ denotes the cardinality of a set F.

Roughly speaking, we can choose a finite subset in an amenable group having small boundary compared with its cardinality (see §1). This property is quite useful in the study of the spectral theory of complete Riemannian manifolds admitting an amenable group action with compact orbit space ([1], [3]). In this note we refine this characterization from a geometric point of view, precisely, a viewpoint

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