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Nagoya Math. J.
Vol. 139 (1995), 93–128

GLOBAL SOLUTION FOR THE YANG–MILLS GRADIENT FLOW ON 4–MANIFOLDS

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1. Introduction

In this paper, we will study a global weak solution for the Yang-Mills gradient flow on a closed (i.e., compact without boundary) 4-manifold. Let us explain some notion briefly to be able to state our results.

Let M be a closed 4-manifold, G a compact Lie group embedded as a subgroup of $SO(l)$, or $SU(l)$ and P be a principal G-bundle over M. We now assume the universal covering $\tilde{G}$ of G is compact. Denote by $\mathfrak{g}$ the Lie algebra of G and denote also by $\mathfrak{g}_P$ and $\mathfrak{G}_P$ the adjoint and automorphism bundles of P, respectively. Using the metric on G induced by the Killing form, we fix a metric on P compatible with the action of G. Let $\Omega^k(\mathfrak{g}_P)$ be the space of smooth $\mathfrak{g}$-valued k-forms, i.e., $\Omega^k(\mathfrak{g}_P) = C^\infty(M; \mathfrak{g}_P \otimes \wedge^k T^*M)$. Here, for the space $\Omega^k(\mathfrak{g}_P)$ of $\mathfrak{g}_P$-valued k-forms, we can define Sobolev spaces $W^{m,p}$, L^p with norms $\| \ \|_{W^{m,p}}$, $\| \ \|_p$ in usual way.

Connections on P are explained by taking an open covering $\{U_\alpha\}$ on M; we trivialize P on U_α via a trivialization: $P|_{U_\alpha} \cong U_\alpha \times G$. A connection D on P is, by definition, given by $D = d + A_\alpha$ on U_α, where A_α is a $\mathfrak{g}$-valued 1-form on U_α. Moreover, for a set of transition functions $\{g_{\alpha\beta}\}$ of P associated with the trivialization for $\{U_\alpha\}$, where $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \to G$, D satisfies

$$A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta.$$

We denote by d_D and d_D^* the covariant exterior differentiation and its formal adjoint with respect to a connection D, respectively. Moreover, the covariant differentiation on tensors for the connection D is defined by $\tilde{\nabla}_D$. If D is a smooth connection, then its curvature is given by $R_D = d_D^{\,2} \in \Omega^2(\mathfrak{g}_P)$.

We consider the Yang-Mills gradient flow; the steepest descent flow of the

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Received December 11, 1992.
Revised September 14, 1994.
†Partially supported by the Ishida Foundation.