H. Azad, J. J. Loeb and M. N. Qureshi
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TOTALLY REAL ORBITS IN AFFINE QUOTIENTS OF REDUCTIVE GROUPS

H. AZAD, J. J. LOEB AND M. N. QURESHI

Let K be a compact connected Lie group and L a closed subgroup of K. In [8] M. Lassalle proves that if K is semisimple and L is a symmetric subgroup of K then the holomorphy hull of any K-invariant domain in $K^{\mathbf{C}}/L^{\mathbf{C}}$ contains K/L. In [1] there is a similar result if L contains a maximal torus of K. The main group theoretic ingredient there was the characterization of K/L as the unique totally real K-orbit in $K^{\mathbf{C}}/L^{\mathbf{C}}$. On the other hand, Patrizio and Wong construct in [9] special exhaustion functions on the complexification of symmetric spaces K/L of rank 1 and find that the minimum value of their exhaustions is always achieved on K/L. By a lemma of Harvey and Wells [6] one knows that the set where a strictly plurisubharmonic (briefly s.p.s.h) function achieves its minimum is totally real. There is a related result in [2, Lemma 1.3] which states that if ϕ is any differentiable function on a complex manifold M then the form $dd^{\mathbf{C}}\phi$ vanishes identically on any real submanifold N contained in the critical set of ϕ; in particular if φ is s.p.s.h then N must be totally real. In view of these results we give in this note a description of all totally real K-orbits in the affine quotients $K^{\mathbf{C}}/L^{\mathbf{C}}$ of K/L. Our main result is as follows:

PROPOSITION. *Let $G = K^{\mathbf{C}}$, $H = L^{\mathbf{C}}$. The group L has finitely many totally real orbits in G/H if and only if $N(H^{\circ})/H^{\circ}$ is finite, H° being the connected component of H and $N(H^{\circ})$ its normalizer in G, and in this case there is a unique totally real K-orbit in G/H.*

This proposition has the following consequence.

COROLLARY. *If $N(H^{\circ})/H^{\circ}$ is finite then any K-invariant s.p.s.h. function on G/H is proper and achieves its minimum value on K/L. Moreover, the holomorphy hull of any K-invariant domain in G/H meets K/L.*

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