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CLASSICAL SOLUTIONS OF THE THIRD PAINLEVÉ EQUATION

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1. Introduction and main results

The big problem "Do Painlevé equations define new functions ? ", what is called the problem of irreducibilities of Painlevé equations, was essentially solved by H. Umemura [16], [17] and K. Nishioka [9].

Umemura [16] analyzed Painlevé's Stockholm Lessons [15] and extracted the concept of "classical functions". To define "classical functions", Umemura introduced the permissible operations to construct new known functions from already known functions. First, we note that we identify a holomorphic function f on an open set $U \subset \mathbf{C}$ with its restriction $f|_V$ onto an open subset $V \subset U$. Let S be a certain set of meromorphic functions on a domain $D \subset \mathbf{C}$. We assume that all the elements in S are already known functions. Permissible operations to construct new known functions from the set S are as follows.

Definition I [16, Part II §2]. (O) Let $f(t) \in S$. Then the derived function $f'(t)$ is a new known function.
(P1) If $f_1, f_2 \in S$, then the sum $f_1 + f_2$ and the product $f_1 f_2$ are new known functions. Moreover if $f_2 \neq 0$, then the quotient f_1/f_2 is a new known function.
(P2) Let $a_1, \cdots, a_n \in S$. Then any solution f of an algebraic equation $f^n + a_1 f^{n-1} + \cdots + a_n = 0$ is a new known function.
(P3) Let $f(t) \in S$. Then the quadrature $\int f(t)\,dt$ is a new known function.
(P4) Let $a_1, \cdots, a_n \in S$. Then any solution f of a linear differential equation $d^n f/dt^n + a_1 d^{n-1} f/dt^{n-1} + \cdots + a_n f = 0$ is a new known function.
(P5) Let $\Gamma \subset \mathbf{C}^n$ be a lattice such that the quotient $\mathbf{C}^n/\Gamma$ is an abelian variety. Let $\pi : \mathbf{C}^n \to \mathbf{C}^n/\Gamma$ be the projection. Let $f_1, \cdots, f_n \in S$ be holomorphic functions on a domain $D \subset \mathbf{C}$ and ϕ be a meromorphic function on $\mathbf{C}^n/\Gamma$. Then the function $\phi \cdot \pi \cdot (f_1, \cdots, f_n)$ is a new known function if it is not the constant function taking

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