C. Sabbah
Nagoya Math. J.
Vol. 141 (1996), 107–124

ON THE COMPARISON THEOREM FOR ELEMENTARY IRREGULAR $\mathcal{D}$-MODULES

CLAUDE SABBAH

Introduction

Let U be a smooth quasi-projective variety over $\mathbf{C}$ and let f be a regular function on U. Let $\mathcal{D}_U$ be the sheaf of algebraic differential operators on U and let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_U$-module: here, regular means that there exists some smooth compactification X of U and some extension of $\mathcal{M}$ as a $\mathcal{D}_X$-module which is regular holonomic on X (one also may avoid the use of a *smooth* compactification to define regularity, see [17]).

Let $\mathcal{M}_f$ be the $\mathcal{D}_U$-module obtained from $\mathcal{M}$ by twisting by e^f. By definition, $\mathcal{M}_f$ is equal to $\mathcal{M}$ as an $\mathcal{O}_U$-module; the operator $\nabla_f : \mathcal{M}_f \to \Omega_U^1 \otimes_{\mathcal{O}_U} \mathcal{M}_f$ is equal to $e^{-f} \nabla e^f$, where ∇ is the operator $\mathcal{M} \to \Omega_U^1 \otimes_{\mathcal{O}_U} \mathcal{M}$ given by the $\mathcal{D}_U$-module structure; we have $\nabla_f^2 = 0$ because $\nabla^2 = 0$ and this defines a $\mathcal{D}_U$-module structure on $\mathcal{M}_f$.

Let $\mathrm{DR}(\mathcal{N})$ be the algebraic de Rham complex of the holonomic $\mathcal{D}_U$-module $\mathcal{N}$:

$$(*) \qquad \mathrm{DR}(\mathcal{N}) = \{0 \to \mathcal{N} \xrightarrow{\nabla} \Omega_U^1 \otimes_{\mathcal{O}_U} \mathcal{N} \xrightarrow{\nabla} \Omega_U^2 \otimes_{\mathcal{O}_U} \mathcal{N} \xrightarrow{\nabla} \cdots\}$$

(it is now usual to consider that the term corresponding to $\Omega^{\dim U}$ is in degree 0, but it will not matter here and we shall not shift this complex). We shall give a formula for the hypercohomology of $\mathrm{DR}(\mathcal{M}_f)$, *i.e.* the cohomology of the complex $R\Gamma(U, \mathrm{DR}(\mathcal{M}_f))$. If U is affine, this is the cohomology of the complex $\mathrm{DR}(\mathcal{M}_f(U))$ of global sections over U.

This result was conjectured in [1] in a particular case, where U is the complement of an arrangement of hyperplanes in general position in $\mathbf{C}^l$ and $\mathcal{M}$ is a rank one locally free $\mathcal{O}_U$-module.

In fact, the global comparison theorem we give is essentially equivalent to the one given in [8] (see also [15] and [22]).

We shall use this result to obtain vanishing theorems of the type given in [1]

Received October 18, 1994.