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THE MINIMUM AND THE PRIMITIVE REPRESENTATION OF POSITIVE DEFINITE QUADRATIC FORMS II

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We are concerned with representation of positive definite quadratic forms by a positive definite quadratic form. Let us consider the following assertion

$\mathrm{A}_{m,n}$: *Let M, N be positive definite quadratic lattices over $\mathbf{Z}$ with $\mathrm{rank}(M) = m$ and $\mathrm{rank}(N) = n$ respectively. We assume that the localization M_p is represented by N_p for every prime p, that is there is an isometry from M_p to N_p. Then there exists a constant $c(N)$ dependent only on N so that M is represented by N if $\min(M) > c(N)$, where $\min(M)$ denotes the least positive number represented by M.*

We know that the assertion $A_{m,n}$ is true if $n \geq 2m + 3$. A suceeding natural problem is whether it is the best or not. It is known that this is the best if $m = 1$, that is $\mathrm{A}_{1,4}$ is false. But in the case of $m \geq 2$, what we know at present, is that there is an example N so that $\mathrm{A}_{m,n}$ is false if $n - m = 3$. We do not know such examples when $n - m = 4$. Anyway, analyzing the counter-example, we come to the following two assertions $\mathrm{APW}_{m,n}$ and $\mathrm{R}_{m,n}$.

$\mathrm{APW}_{m,n}$: *There exists a constant $c'(N)$ dependent only on N so that M is represented by N if $\min(M) > c'(N)$ and M_p is primitively represented by N_p for every prime p.*

$\mathrm{R}_{m,n}$: *There is a lattice M' containing M such that M'_p is primitively represented by N_p for every prime p and $\min(M')$ is still large if $\min(M)$ is large.*

If the assertion $\mathrm{R}_{m,n}$ is true, then the assertion $\mathrm{A}_{m,n}$ is reduced to the apparently weaker assertion $\mathrm{APW}_{m,n}$. If the assertion $\mathrm{R}_{m,n}$ is false, then it becomes possible to make a counter-example to the assertion $\mathrm{A}_{m,n}$. As a matter of fact, $\mathrm{APW}_{1,4}$ is true but $\mathrm{R}_{1,4}$ is false in general, and it yields examples of N such that $\mathrm{A}_{1,4}$ is false.

We proved that the assertion $\mathrm{R}_{m,2m+1}$ (resp. $\mathrm{R}_{m,2m+2}$) is true if $m \geq 3$ (resp. $m \geq 2$), respectively. The aim of this paper is study the case of $n = 2m$ for $m \geq 4$. In Section 1, we study $\min \sum_{i=1}^{t} \lceil br_i/N \rfloor^2 q_i$ where q_i is a positive number, r_i, N are integers, b runs over integers $\not\equiv 0 \bmod N$ and $\lceil x \rfloor$ $(-0.5 \leq \lceil x \rfloor < 0.5)$

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