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ALGEBRAIC BARTH-LEFSCHETZ THEOREMS

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0. Introduction

We shall work over a fixed algebraically closed field k of arbitrary characteristic. By an algebraic variety over k we shall mean a reduced algebraic scheme over k. Fix a positive integer n and $e = (e_0, e_1, \dots, e_n)$ a system of $n + 1$ weights (i.e. $n + 1$ positive integers $e_0, e_1, \dots, e_n$). If $k[T_0, T_1, \dots, T_n]$ is the polynomial k-algebra in $n + 1$ variables, graded by the conditions $\deg(T_i) = e_i$, $i = 0,1,\dots, n$, denote by $\mathbf{P}^n(e) = \mathrm{Proj}(k[T_0, T_1, \dots, T_n])$ the n-dimensional weighted projective space over k of weights e. We refer the reader to [3] for the basic properties of weighted projective spaces. According to Zariski ([22], see also [16], [15]), if Y is a closed subscheme of an algebraic variety X, one can define the ring $K(\hat{X}_{/Y})$ of formal rational functions of X along Y. Then $K(\hat{X}_{/Y})$ is a k-algebra, and there is a canonical map of k-algebras $K(X) \to K(\hat{X}_{/Y})$, where $K(X)$ is the usual ring of rational functions of X ($K(X)$ is a field if X is irreducible). According to [16], Y is said to be $\mathbf{G}_3$ in X if this map is an isomorphism. Let X be an arbitrary algebraic scheme over k, and let $d \geq 0$ be a non-negative integer. Then X is said to be d-connected if every irreducible component of X is of dimension $\geq d + 1$ and if $X \setminus W$ is connected for every closed subscheme W of X of dimension $< d$. For example, X is 0-connected if X is connected and of dimension ≥ 1; an irreducible algebraic variety X of dimension $n \geq 1$ is always $(n - 1)$-connected.

Then the main result of this paper is the following.

THEOREM (0.1). *Let $f : X \to \mathbf{P}^n(e) \times \mathbf{P}^n(e)$ be a finite morphism from a d-connected algebraic variety X such that $d \geq n$. Then $f^{-1}(\Delta)$ is $(d - n)$-connected, where Δ is the diagonal of $\mathbf{P}^n(e) \times \mathbf{P}^n(e)$. Moreover, $f^{-1}(\Delta) \setminus W$ is $\mathbf{G}_3$ in $X \setminus W$ for every closed subscheme W of $f^{-1}(\Delta)$ of dimension $< d - n$.*

In the case of ordinary projective spaces (i.e. when $e_i = 1$ for every $i = 0, 1, \dots, n$) the first statement of Theorem (0.1) is well known in the literature as

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