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HYPONORMAL TOEPLITZ OPERATORS ON $H^2(T)$ WITH POLYNOMIAL SYMBOLS

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Let T be the unit circle on the complex plane, $H^2(T)$ be the usual Hardy space on T, T_ϕ be the Toeplitz operator with symbol $\phi \in L^\infty(T)$, C. Cowen showed that if f_1 and f_2 are functions in H^2 such that $f = f_1 + \bar{f}_2$ is in L^∞, then T_f is hyponormal if and only if $f_2 = c + T_{\bar{g}} f_1$ for some constant c and some function g in H^∞ with $\|g\|_\infty \leq 1$ [1]. Using it, T. Nakazi and K. Takahashi showed that the symbol of hyponormal Toeplitz operator T_ϕ satisfies $\phi - g = k\bar{\phi}$, $g \in H^\infty$ and $k \in H^\infty$ with $\|k\| \leq 1$ [2], and they described the ϕ solving the functional equation above. Both of their conditions are hard to check, T. Nakazi and K. Takahashi remarked that even "the question about polynomials is still open" [2]. Kehe Zhu gave a computing process by way of Schur's functions so that we can determine any given polynomial ϕ such that T_ϕ is hyponormal [3]. Since no closed-form for the general Schur's function is known, it is still valuable to find an explicit expression for the condition of a polynomial ϕ such that T_ϕ is hyponormal and depends only on the coefficients of ϕ, here we have one, it is elementary and relatively easy to check. We begin with the most general case and the following Lemma is essential.

LEMMA 1. *If $f, g \in H^2(T)$ and $\phi = f + \bar{g} \in L^\infty(T)$, then T_ϕ is hyponormal if and only if the (bounded) operator A on l^2*

$$(1) \qquad \begin{aligned} A = (A_{ij}) &\equiv (A_f(i, j) - A_g(i, j)) \\ &\equiv (\langle S^{*^j} f, S^{*^i} f\rangle - \langle S^{*^j} g, S^{*^i} g\rangle) \quad i, j \geqq 1 \end{aligned}$$

is non-negative where S refers to the unilateral shift on $H^2(T)$.

Proof. By definition T_ϕ is hyponormal when $T_\phi^* T_\phi - T_\phi T_\phi^* \geqq 0$, i.e. $(T_{f+\bar{g}})^* T_{f+\bar{g}} - T_{f+\bar{g}}(T_{f+\bar{g}})^* = (T_f^* T_f - T_f T_f^*) - (T_g^* T_g - T_g T_g^*) \geqq 0$, the Lemma

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