F. Chamizo and H. Iwaniec
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ON THE GAUSS MEAN-VALUE FORMULA FOR CLASS NUMBER

FERNANDO CHAMIZO AND HENRYK IWANIEC

Abstract. In his masterwork *Disquisitiones Arithmeticae*, Gauss stated an approximate formula for the average of the class number for negative discriminants. In this paper we improve the known estimates for the error term in Gauss approximate formula. Namely, our result can be written as $N^{-1}\sum_{n\le N} H(-n) = 4\pi\sqrt{N}/(21\zeta(3)) - 2\pi^2 + O(N^{-15/44+\epsilon})$ for every $\epsilon > 0$, where $H(-n)$ is, in modern notation, $h(-4n)$. We also consider the average of $h(-n)$ itself obtaining the same type of result.

Proving this formula we transform firstly the problem in a lattice point problem (as probably Gauss did) and we use a functional equation due to Shintani and Dirichlet class number formula to express the error term as a sum of character and exponential sums that can be estimated with techniques introduced in a previous work on the sphere problem.

§1. Introduction and statement of the main result

Gauss noted in Art. 302 of [Ga] that the average of the class number for negative discriminants increases very regularly, and he gave (without proof) an approximate formula for this average, checking its accuracy up to unbelievably large values. (He, in fact, computed not only the class number but also the subdivision into genera for several thousands of discriminants).

Probably Gauss proved that his formula gives the correct asymptotics but the first known proof of this fact is due to Lipschitz in 1865. In this century I. M. Vinogradov studied the error term in Gauss' approximation using the method of trigonometric sums. In a series of papers, along fifty years, he gave several upper bounds that also hold for the "sphere problem" (the three-dimensional analogue of the circle problem).

In [Ch-Iw] we improved the results of Chen [Ch] and Vinogradov [Vi] on the sphere problem. The key point in our method is the double interpretation of the sphere problem; the first one leads us to trigonometric sums and the second one to character sums. The purpose of this paper is to apply

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