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DEGREE BOUNDS FOR GENERATORS OF COHOMOLOGY MODULES AND CASTELNUOVO-MUMFORD REGULARITY

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Abstract. By extending Mumford's result on the generating by global sections there are estimates on the degree for generators of local cohomology modules. These arguments provide bounds on the Castelnuovo-Mumford regularity, in particular for Cohen-Macaulay varieties. As an application they imply a few more cases of varieties that satisfy a conjecture posed by Eisenbud and Gôto.

§1. Introduction

Let $\mathcal{F}$ denote a coherent sheaf on the projective space $\mathbb{P}^n = \mathbb{P}^n_K$, K denotes an algebraically closed field. In [15], Lecture 14, $\mathcal{F}$ is called m-regular, $m \in \mathbb{Z}$, provided $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$ for all $i > 0$. Then it turns out, see loc. cit., that $\mathcal{F}(k)$ is generated as $\mathcal{O}_{\mathbb{P}^n}$-module by its global sections if $k \geq m$. By more recent results, see e. g. [5], this is generalized to the generation of $\mathcal{S}_j$, the j-th sheaf of syzygies of $\mathcal{F}$. Here we want to show another generalization of Mumford's result. In order to formulate our approach we fix a few notation. For $s > 0$ let

$$r_s(\mathcal{F}) := \min\{m \in \mathbb{Z} \mid H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0 \text{ for all } i \geq s\}.$$

Note that $\operatorname{reg} \mathcal{F} = r_1(\mathcal{F})$ is called the Castelnuovo-Mumford regularity of $\mathcal{F}$. Hence $\mathcal{F}$ is m-regular for all $m \geq \operatorname{reg} \mathcal{F}$. Furthermore, define $e_i^+(\mathcal{F})$ the smallest integer $m \in \mathbb{Z}$ such that $H^i(\mathbb{P}^n, \mathcal{F}(k))$ is spanned by $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^i(\mathbb{P}^n, \mathcal{F}(k-1))$ for all $k > m$. By Serre's vanishing result this is true for all $m \gg 0$. More precisely, Mumford's result, see loc. cit., says $e_0^+(\mathcal{F}) \leq \operatorname{reg} \mathcal{F}$. Its extension is our first main result.

THEOREM 1.1. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Then there is the following bound*

$$e_i^+(\mathcal{F}) \leq r_{i+1}(\mathcal{F}) - i$$

for all $i \geq 0$.

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