ON THE CANONICAL FORM OF TURBULENCE

SEIZÔ ITÔ

§ 0. **Introduction.** In K. Itô's paper [1] on the theory of turbulence, the problem to determine the canonical form of the moment tensor of temporally homogeneous and isotropic turbulence, has not been solved. In the present paper, the author will solve the problem by making use of the result of his preceding paper [2]. We shall treat the turbulence in R^3, but the similar argument is possible in R^n.

§ 1. **Generalities.** In the theory of *turbulence*, the deviation of the velocity from its mean may be considered as a system of random vectors $\mathfrak{u}(t, \mathfrak{X}, \omega) = \langle u_p(t, \mathfrak{X}, \omega)/p = 1, 2, 3\rangle$, where $t \in R^1$ and $\mathfrak{X} \in R^3$ denote the time and the position respectively and $\omega \in (\Omega, P)$ is the probability parameter; we assume naturally that $u_p(t, \mathfrak{X}, \omega)$ is B-measurable in $\langle t, \mathfrak{X}, \omega\rangle$ and belongs to $L^2(\Omega, \mathrm{P})$ for any fixed $\langle p, t, \mathfrak{X}\rangle$.[1] We have clearly

$$(1.1)\qquad \mathbf{E}_\omega[u_p(t, \mathfrak{X}, \omega)] = 0 \quad (\mathbf{E}_\omega[.]\text{ denotes the expectation}).$$

Now we define the moment tensor of the turbulence by

$$(1.2)\qquad R_{pq}(t, \mathfrak{X};\ s, \mathfrak{Y}) = \mathbf{E}_\omega[u_p(t, \mathfrak{X}, \omega)\, u_q(s, \mathfrak{Y}, \omega)];$$

then

$$(1.3)\qquad R_{pq}(t, \mathfrak{X};\ s, \mathfrak{Y}) = R_{qp}(s, \mathfrak{Y};\ t, \mathfrak{X}), \text{ and}$$

$$(1.4)\qquad \textstyle\sum_{i,j}\alpha_i\bar{\alpha}_j R_{p_i p_j}(t_i, \mathfrak{X}_i;\ t_j, \mathfrak{X}_j) \geqq 0^{2)} \quad (\alpha_i:\text{ complex number}).$$

We consider the turbulence satisfying the following three conditions:

$$(1.5)\qquad R_{pq}(t+\tau, \mathfrak{X};\ s+\tau, \mathfrak{Y}) = R_{pq}(t, \mathfrak{X};\ s, \mathfrak{Y}) \quad (\textit{temporally homogeneous});$$

$$(1.6)\qquad R_{pq}(t, \mathfrak{X}+\mathfrak{a};\ s, \mathfrak{Y}+\mathfrak{a}) = R_{pq}(t, \mathfrak{X};\ s, \mathfrak{Y}) \quad (\textit{spatially homogeneous});$$

and

$$(1.7)\qquad \textstyle\sum_{p',q'} k_{p'p}k_{q'q}R_{p'q'}(t, \mathfrak{X};\ s, \mathfrak{X}+K(\mathfrak{Y}-\mathfrak{X})) = R_{pq}(t, \mathfrak{X};\ s, \mathfrak{Y})$$

for any orthogonal transformation $K \equiv (K_{pq}/p,\ q = 1, 2, 3)$ (*isotropic*). We can easily prove by (1.3) that the isotropism implies the homogenuity (1.6). Consequently we get

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[1] See [1].

[2] If $R_{pq}(t, \mathfrak{X};\ s, \mathfrak{Y})$ is real-valued and satisfies (1.4), then it satisfies (1.3) automatically.