ON EVERYWHERE DENSE IMBEDDING OF FREE GROUPS IN LIE GROUPS

MASATAKE KURANISHI

In this note it will be proved that some kinds of Lie groups (including semi-simple Lie groups) have an everywhere dense subgroup which is algebraically isomorphic to the free group generated by two elements (Theorem 8).

In § 1 characterizations of Lie groups which are approximated by discrete subgroups[1] are given. This section is closely connected with a part of the results of Malcev [4] and Matsushima [5], and some theorems are slight modifications of them (Theorems 2 and 3).

In § 2 a sufficient condition for Lie algebras to be generated by two elements is given, and in § 3 the main theorem is proved.

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§ 1. Lie groups approximated by discrete subgroups

THEOREM 1. *Let G be an n-dimensional local Lie group and U a neighborhood of the identity e of G, in which a canonical coordinates system is introduced. Let H be a discrete subset of G satisfying the following conditions.*

1) *If V is any neighborhood of e such that $VV^{-1} \subset U$, then for any $x, y \in V \cap H$, $xy^{-1} \in H$.*

2) *H contains $h_1, h_2, \ldots, h_n$, which are linearly independent in U (with respect to the coordinates system in U).*

Then G is a nilpotent Lie group.

Proof. It is easily seen that if x and y are elements of U, then

(1) $$|xyx^{-1}y^{-1}| \lneqq \min(|x|, |y|),$$

where $|x|$ is the euclidean distance between e and x in U. Let p be the point of H which is not equal to e and $|p| \leqq |x|$ for every $x \in H$. Then by (1) p is commutative with every element of H, in particular

$$h_i p h_i^{-1} = p \qquad (i = 1, 2, \ldots, n).$$

Hence

$$h_i p^\lambda h_i^{-1} = p^{\lambda}\,{}^{[2]} \qquad (i = 1, 2, \ldots, n),$$

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[1] Cf. Definition 1. This notion is introduced by H. Tôyama [7].

[2] Let x^λ be the one-parameter subgroup passing x such that $x^1 = x$. We use this notation throughout this note.