NOTE ON SUBDIRECT SUMS OF RINGS

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In my previous paper "On the theory of semi-local rings,"[1] we saw that if a semi-local ring R with maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ is a subdirect sum of local rings $R_{[\mathfrak{p}_i]}$,[2] then R is the direct sum of $R_{[\mathfrak{p}_i]}$ (proposition 15, (slr)[1]) and that a complete semi-local ring is a direct sum of complete local rings (Remark to proposition 5, (slr)).

The main purpose of the present note is to prove two kinds of generalization (also for non-commutative case) of the first assertion mentioned above (Theorems 2 and 3). We first introduce in § 1 the concept of n-rings and then we define the concepts of semi-local rings, local rings and so on; it is proved here that a commutative (semi-) local ring is a (semi-) local ring in the sense of (slr). It is also remarked that the assumption in Proposition 15, (slr), is a necessary and sufficient condition in order that a commutative semi-local ring is a direct sum of local rings. In § 2, we prove our main theorems. In § 3, we prove a generalization of the second assertion mentioned above for non-commutative case; in § 4 we study rings which are subdirect sums of (a finite number of) n-rings.

1. Definitions and remarks to commutative case

DEFINITION 1. A ring[3] R is called an n-ring if $R^2 = R$ and if for any proper ideal[4] $\mathfrak{a}$ in R there exists a maximal ideal[5] containing $\mathfrak{a}$.

DEFINITION 2. A quasi-semi-local ring is a non-zero n-ring which contains only a finite number of maximal ideals. A quasi-local ring is a non-zero n-ring which contains only one maximal ideal.

DEFINITION 3. A quasi-semi-local ring R with maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ is called a semi-local ring if $\bigcap_{i,n} \mathfrak{p}_i^n = (0)$. In this case we introduce a topology in R by taking $\{\bigcap_{i=1}^{h} \mathfrak{p}_i^n ; n = 1, \ldots, k, \ldots\}$ as a system of neighbour-

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1) To appear in Proc. Jap. Acad. and will be referred as (slr) in the present note.

2) This notation is same as in (slr); this denotes the topological quotients ring of $\mathfrak{p}_i$ with respect to R: See Chapter I, (slr).

3) A ring means an associative ring.

4) An ideal means a two-sided ideal.

5) Since $R^2 = R$, any maximal ideal is prime (we say an ideal $\mathfrak{p}$ in a ring R is maximal if $R \neq \mathfrak{p}$ and if there exists no ideal $\mathfrak{a}$ such as $R \supset \mathfrak{a} \supset \mathfrak{p}$).