NOTE ON p-GROUPS

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In connection with the class field theory a problem concerning p-groups was proposed by W. Magnus [1]: Is there any infinite tower of p-groups $G_1, G_2, \dots, G_n, G_{n+1}, \dots$ such that G_1 is abelian and each G_n is isomorphic to $G_{n+1}/\theta_n(G_{n+1})$, $\theta_n(G_{n+1}) \neq 1$, $n = 1, 2, \dots$, where $\theta_n(G_{n+1})$ denotes the n-th commutator subgroup of G_{n+1}? The present note [2] is, firstly, to construct indeed such a tower, to settle the problem, and also to refine an inequality for p-groups of P. Hall.[3]

1. Let p be an odd prime number and let M_i be the principal congruence subgroup of "stufe" (p^i) of the homogeneous modular group in the rational p-adic number field R_p, that is, the totality of matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ such that a_{11}, $a_{12}, a_{21}, a_{22} \in R_p$, $a_{11} \equiv a_{22} \equiv 1 \pmod{p^i}$, and $a_{12} \equiv a_{21} \equiv 0 \pmod{p^i}$. Let $\theta_r(M_i)$ denote the r-th commutator subgroup of M_i.

LEMMA 1. $\theta_s(M_i) \leqq M_{2^s}$ for $s = 0, 1, 2, \dots$.

Proof. The case $s = 0$ is trivial. Assume $s > 0$ and that $\theta_{s-1}(M_1) \leqq M_{2^{s-1}}$. Then $\theta_s(M_1) \leqq \theta_1(M_{2^{s-1}})$. We shall prove $\theta_1(M_{2^{s-1}}) \leqq M_{2^s}$.
Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be any two elements of $M_{2^{s-1}}$. Then

$$A^{-1}B^{-1}AB = |A|^{-1} \cdot |B|^{-1}$$
$$\begin{pmatrix} (a_{22}b_{22} + a_{12}b_{21})(a_{11}b_{11} + a_{12}b_{21}) - (a_{22}b_{12} + a_{12}b_{11})(a_{21}b_{11} + a_{22}b_{21}) & (a_{22}b_{22} + a_{12}b_{21})(a_{11}b_{12} + a_{12}b_{22}) - (a_{22}b_{12} + a_{12}b_{11})(a_{21}b_{12} + a_{22}b_{22}) \\ -(a_{21}b_{22} + a_{11}b_{21})(a_{11}b_{11} + a_{12}b_{21}) + (a_{21}b_{12} + a_{11}b_{11})(a_{21}b_{11} + a_{22}b_{21}) & -(a_{21}b_{22} + a_{11}b_{21})(a_{11}b_{12} + a_{12}b_{22}) + (a_{21}b_{12} + a_{11}b_{11})(a_{21}b_{12} + a_{22}b_{22}) \end{pmatrix}$$

where $|A|, |B|$ are the determinants of A, B respectively, and therefore $|A|^{-1}a_{11}a_{22} \equiv |B|^{-1}b_{11}b_{22} \equiv 1 \pmod{p^{2^s}}$. Now $a_{11} \equiv a_{22} \equiv b_{11} \equiv b_{22} \equiv 1 \pmod{p^{2^{s-1}}}$, $a_{12} \equiv a_{21} \equiv b_{12} \equiv b_{21} \equiv 0 \pmod{p^{2^{s-1}}}$. Then $(1,1)$- and $(2,2)$-elements of $A^{-1}B^{-1}AB$ are obviously $\equiv 1 \pmod{p^{2^s}}$. Since

$$a_{22}b_{22}(a_{11}b_{12} + a_{12}b_{22}) - (a_{22}b_{12} + a_{12}b_{11})a_{22}b_{22} = a_{22}b_{22}\{b_{12}(a_{11} - a_{22}) + a_{12}(b_{22} - b_{11})\},$$
$$-(a_{21}b_{22} + a_{11}b_{21})a_{11}b_{11} + a_{11}b_{11}(a_{21}b_{11} + a_{22}b_{21}) = a_{11}b_{11}\{a_{21}(b_{11} - b_{22}) + b_{21}(a_{22} - a_{11})\},$$

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[1] W. Magnus, Beziehung zwishen Gruppen und Idealen in einem speziellen Ring, Math. Annalen **111** (1935).

[2] An impulse was given to the present work by Dr. K. Iwasawa, through a communication by Mr. M. Suzuki.