ON TITCHMARSH-KODAIRA'S FORMULA CONCERNING WEYL-STONE'S EIGENFUNCTION EXPANSION

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1. Introduction. Let $q(x)$ be real and continuous in the infinite open interval $(-\infty, \infty)$ and let $y_1(x, \lambda)$, $y_2(x, \lambda)$ be the solutions of

$$y'' + \{\lambda - q(x)\}y = 0 \tag{1.1}$$ [1]

with the initial conditions

$$y_1(0,\lambda) = 1, \quad y_1'(0,\lambda) = 0, \quad y_2(0,\lambda) = 0, \quad y_2'(0,\lambda) = 1. \tag{1.2}$$

For appropriate homogeneous real boundary conditions at $x = -\infty$, $x = \infty$ of the differential operator

$$L_x = q(x) - \frac{d^2}{dx^2}, \tag{1.3}$$

there corresponds real symmetric positive definite matrix

$$P(u_2) - P(u_1) = (p_{jk}(u_2) - p_{jk}(u_1)), \quad (j, k = 1, 2), \quad -\infty < u_1 < u_2 < \infty, \tag{1.4}$$

such that we have Weyl[2]-Stone's[3] expansion (in the sense of L_2-convergence):

(1.5) for real-valued $f(x) \in L_2(-\infty, \infty)$,

$$f(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} d_u \Big\{ \sum_{j,k=1}^{2} \int_0^u y_j(x,u)\, dp_{jk}(u) \int_{-n}^{n} f(s) y_k(s,u)\, ds \Big\}.$$

Recently and independently of each other, E. C. Titchmarsh[4] and K. Kodaira[5]

Received December 20, 1949. (Added March 5, 1950). The result was communicated to Prof. K. Kodaira at Princeton, who informed to the author that a similar treatment may be carried on by Prof. N. Levinson. So a copy of the manuscript was sent to Prof. Levinson, who, in his letter of February 25, informed to the author that his work was submitted to the Duke Math. Journal in May, 1949. He says that his method is different from the pressent note; he proceeds in his proof from the Parseval relation of the Sturm-Liouville orthonormal functions.

[1] The case of finite or half finite open interval may be treated exactly in the same manner. Moreover (apparently) general equation $(p(\xi)z_\xi)_\xi + \{\lambda r(\xi) - s(\xi)\}z = 0$ may be reduced to (1.1) by the Liouville Transformation $x = \int_0^\xi (p^{-1}r)^{1/2} d\xi$, $y = (pr)^{1/4} z$.

[2] Über gewöhnlichc Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlichen Funktionen, Math. Ann., 68 (1910), 220-269.

[3] Linear transformations in Hilbert space, Amer. Math. Soc. Coll. Publ. XV (1932).

[4] Eigenfunction expansisons associated with second order differential equations, Oxford (1946).

[5] The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices, Amer. J. of Math., **71** (1949), 921-945.