STOCHASTIC DIFFERENTIAL EQUATIONS IN A DIFFERENTIABLE MANIFOLD

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The theory of stochastic differential equations in a differentiable manifold has been established by many authors from different view-points, especially by P. Lévy [2] [1], F. Perrin [1], A. Kolmogoroff [1] [2] and K. Yosida [1] [2]. It is the purpose of the present paper to discuss it by making use of stochastic integrals.[2]

In § 1 we shall state some properties of stochastic integrals for the later use. We shall discuss stochastic differential equations in the r-dimensional Euclidean space in § 2 and in a differentiable manifold in § 3.

1. Some properties of stochastic integrals. Throughout this note we fix an *r-dimensional Brownian motion* [3]:

(1.1) $\beta(t,\omega) = (\beta^1(t,\omega),\ \beta^2(t,\omega), \dots, \beta^r(t,\omega)), \quad -\infty < t < \infty,$

ω ($\in \Omega$) being the probability parameter with the probability law P and t being the time parameter. We assume that any function of t and ω appearing in this note satisfies the following two conditions:

(1.2) it is measurable in (t, ω),

(1.3) the value it takes at $t = t_0$ is a *B-measurable function* [4] of the joint variable $(\beta(t,\omega),\ \tau \leqq t_0)$ for any t_0.

If it holds

(1.4) $$\xi(s,\omega) - \xi(t,\omega) = \int_t^s a(\tau,\omega)d\tau + \sum_{i=1}^r \int_t^s b_i(\tau,\omega)d\beta^i(\tau,\omega),\ ^{5)}$$
$$u \leqq s \leqq t \leqq v,\ \omega \in \Omega_1 (\subseteqq \Omega),$$

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1) The numbers in [ ] denote those of the references at the end of this paper.

2) K. Itô [1], [3].

3) By an r-dimensional Brownian motion we understand an r-dimensional random process whose components are all one dimensional Brownian motion (Cf. P. Lévy [1] p. 166, § 52, J. L. Doob [1] Theorem 3.9) independent of each other.

4) A mapping f from R^A into R is called to be *B-measurable* if the inverse image of any Borel subset of R by f is also a Borel subset of R^A, that is an element of the least completely additive class that contains all rectangular subsets of R^A.
A random variable $\xi(\omega)$ is called to be a *B-measurable function of the joint variable* $(\xi_\alpha(\omega),\ \alpha \in A)$ if and only if there exists a B-measurable mapping f from R^A into R such that $\xi(\omega) = f(\xi_\alpha(\omega),\ \alpha \in A)$ for every ω. Cf. K. Itô [3] § 1.

5) The sense of this integral is to be understood as a *stochastic integral* introduced by the author. Cf. K. Itô [1], [3] § 7, § 8.