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Self-Dual Metrics on 4-dimensional Circle Bundles

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Abstract

A circle bundle P over an oriented 3-manifold is endowed with a bundle metric g in terms of a connection γ. We investigate the self-duality of g in terms of Yang-Mills condition on γ and base metric curvature conditions, and also verify the circle bundle version of Joyce's theorem with respect to Einstein-Weyl structures together with the generalized monopole solutions.

§1. Introduction Let $\pi : P \longrightarrow M$ be a circle principal bundle over an oriented Riemannian 3-manifold (M, h) and γ a connection on P.

We have then a bundle metric g on P in terms of the base metric h and γ;

$$g = \gamma^2 + \pi^* h. \tag{1}$$

Namely, with respect to this metric the vertical subspace $V_u \cong \mathbf{R}$, $u \in P$, is measured through γ by the standard inner product of $\mathbf{R}$ and the horizontal subspace $H_u \cong T_{\pi(u)}M$ by h such that V_u and H_u are required to be orthogonal.

The subject of this paper is to investigate the self-duality of a bundle metric on an oriented 4-manifold P. Here P is orientable, because $T_uP = V_u \oplus H_u$ has the canonical orientation. So we fix an orientation of P in this way.

Consider a metric g on a general oriented 4-manifold X. We say that g is self-dual when the half part of the Weyl conformal curvature tensor vanishes.

To define the self-duality of metric more precisely, we provide on the bundle $\Omega^2(X)$ of 2-forms the Hodge operator $*$ which is an involution so that $\Omega^2(X)$ decomposes into the ± 1 eigensubbundles $\Omega_\pm$;

$$\Omega^2(X) = \Omega^2_+ \oplus \Omega^2_-. \tag{2}$$