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SIMULTANEOUS EXTENSIONS OF SELBERG INEQUALITY AND HEINZ-KATO-FURUTA INEQUALITY

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ABSTRACT. Based on Heinz-Kato-Furuta inequality, we gave an extension of recent Lin's improvement of a generalized Schwarz inequality in our previous note. We present a simultaneous extension of Selberg and Heinz-Kato-Furuta inequalities. As a consequence, we can sharpen the Heinz-Kato-Furuta inequality and further extensions are obtained by Furuta inequality.

1. Introduction.

Throughout this note, an operator means a bounded linear one acting on a Hilbert space. An operator A is positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. We first cite the Heinz-Kato-Furuta inequality, [8] and also [7]:

The Heinz-Kato-Furuta inequality. *Let A and B be positive operators on H. If an operator T on H satisfies $T^*T \leq A^2$ and $TT^* \leq B^2$, then*

$$|(T|T|^{\alpha+\beta-1}x, y)| \leq \|A^\alpha x\|\|B^\beta y\| \tag{1}$$

for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geq 1$ and $x, y \in H$.

We here remark that the Heinz-Kato inequality is just the case $\alpha+\beta = 1$ in above. Based on (1), we have the following extension of a recent Lin's refinement [11] of the generalized Schwarz inequality. Let $T = U|T|$ be the polar decomposition of an operator T on H in the below.

Theorem A. [2] *Let T be an operator on H and $0 \neq y \in H$. For $z \in H$ satisfying $T|T|^{\alpha+\beta-1}z \neq 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$,*

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2(|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \leq (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y) \tag{2}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and $x \in H$.

It is easily seen that Lin's theorem [11; Theorem 1] is the case $\alpha + \beta = 1$ in Theorem A. As a consequence, we have the following improvement of the Heinz-Kato-Furuta inequality via the Löwner-Heinz inequality, i.e., $A \geq B \geq 0$ implies $A^\alpha \geq B^\alpha$ for $\alpha \in [0,1]$, see [12]:

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