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A GEOMETRICAL STRUCTURE IN THE FURUTA INEQUALITY, II

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ABSTRACT. We discuss some geometrical structures in the Furuta inequality as a continuation of our preceding note. We show the monotonicity of operator functions associated with the Furuta inequality under the chaotic order. Consequently it gives us geometrical views and helps us to explain obtained operator inequalities,

1. Introduction. In what follows, a capital letter means a (bounded linear) operator acting on a Hilbert space H. An operator T is said to be positive, in symbol, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in H$. In particular, we denote by $T > 0$ if T is positive and invertible. The positivity of operators induces the (usual) order $A \geq B$ by $A - B \geq 0$ and moreover the operator monotonicity of $\log t$ does the weaker order $A \gg B$ by $\log A \geq \log B$ for $A, B > 0$. It is called the chaotic order, cf. [7].

It is interesting to discuss order-preserving problems for positive operators. One of the most typical examples is the Löwner-Heinz inequality [18, 22]:

Theorem LH. *If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for $0 \leq \alpha \leq 1$.*

In 1987, Furuta [11] considered a background of Theorem LH and finally proposed it as the following surprising form, which is a historical extension of the Löwner-Heinz inequality:

Theorem F. (The Furuta inequality)
If $A \geq B \geq 0$, then for each $r \geq 0$

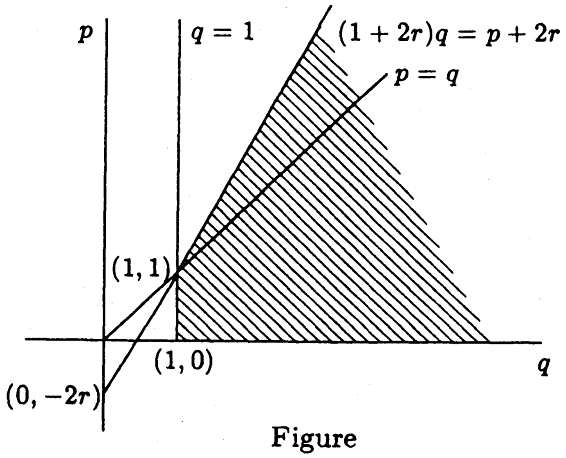
(1) $$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$$

and

(1') $$(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$$

hold for $p \geq 0$ and $q \geq 1$ with

(*) $$(1+2r)q \geq p + 2r.$$



Figure

Related topics are discussed in [2, 3, 9, 12, 13, 14, 15, 17, 19, 20, 23, 24]; among others,an elementary and one-page proof is given in [12] and the best possibility of the conditions on p, q and r in the Furuta inequality is discussed in [23].

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