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PARAMETERIZED KANTOROVICH INEQUALITY FOR POSITIVE OPERATORS

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ABSTRACT. The Kantorovich inequality says that if A is a positive operator on H such that $0 < m \le A \le M$ for some $M \ge m > 0$, then

$$(Ax, x)(A^{-1}x, x) \le \frac{(M+m)^2}{4Mm}$$

for all unit vectors $x \in H$. We generalize it by the use of a family of power means, which gives us a parameterization of the Kantorovich inequality. Moreover we give a parameterization of the Pólya-Szegö inequality.

1. Introduction. Let a, g and h be the arithmetic, geometric and harmonic mean respectively. It is known that these means are unified by the family of power means $\{m_r; -1 \le r \le 1\}$, i.e.,

$$\alpha\ m_r\ \beta = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}} \quad \text{for } \alpha, \beta > 0. \tag{1}$$

It is easily seen that $m_1 = a, m_0 = g$ and $m_{-1} = h$. The family of power means plays an interesting role, e.g., [1,3,5,7]. We refer to [6] for the theory of operator means.

Now Kantorovich established the following inequality in his study on applications of functional analysis to numerical analysis, cf. [2] : If $\{a_k\}$ is a sequence in $\mathbb{R}$ such that $0 < m \le a_k \le M$ for some m and M, then

$$\sum_k a_k x_k^2 \sum_k \frac{1}{a_k} x_k^2 \le \frac{(M+m)^2}{4Mm}\left(\sum_k x_k^2\right)^2$$

holds for all $x = \{x_k\}$ in $l^2(\mathbb{N})$.

If we define the diagonal operator A by $A = \operatorname{diag}(a_k)$, then we have

$$(Ax, x)(A^{-1}x, x) \le \frac{(M+m)^2}{4Mm}\|x\|^4 \quad \text{for } x \in l^2(\mathbb{N})$$

if $0 < m \le A \le M$. As a matter of fact, the following inequality is proved by Greub and Rheinboldt [2], which we call the Kantorovich inequality.

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