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NORM INEQUALITIES RELATED TO MCINTOSH TYPE INEQUALITY

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ABSTRACT. We consider norm inequalities associated to McIntosh type inequality $\|AB\| \le \|\mathrm{Re}\, BA\|$ which is closely related to the Heinz one. Consequently, we give a simple and elementary proof of the Heinz inequality.

1. Introduction. This work is a continuation of preceding paper [2] in some sense. Throughout this note, a capital letter means a (bounded linear) operator on a Hilbert space. Our starting point is the following norm inequality due to Heinz [7]:

Theorem A. *If A and B are positive operators, then*

$$\|AQ + QB\| \ge \|A^r QB^{1-r} + A^{1-r}QB^r\| \tag{1}$$

for $0 \le r \le 1$.

To give an elementary proof to the Heinz inequality, McIntosh [9] showed the following inequality which is just the case $r = 1/2$ in Theorem A.

Theorem B. *For arbitrary operators P, Q and R,*

$$\|P^*PQ + QRR^*\| \ge 2\|PQR\|. \tag{2}$$

Very recently, we pointed out in [2] that both inequalities (1) and (2) are equivalent to an interesting inequality recently obtained by Corach, Porta and Recht [1] that

$$\|STS^{-1} + S^{-1}TS\| \ge 2\|T\| \tag{3}$$

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