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REVERSE INEQUALITIES OF ARAKI, CORDES AND LÖWNER-HEINZ INEQUALITIES

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ABSTRACT. In this paper, we show reverse inequality to Araki's inequality and investigate the equivalence among reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities. Among others, we show that if A and B are positive operators on a Hilbert space H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$K(m,M,p)\|BAB\|^p \leq \|B^pA^pB^p\| \qquad \text{for all } 0 < p < 1,$$

where $K(m,M,p)$ is a generalized Kantorovich constant by Furuta.

1. INTRODUCTION

Let A and B be positive operators on a Hilbert space H. The equivalence among Cordes and Löwner-Heinz inequalities was discussed by many authors. In [10], Furuta showed that the Cordes inequality for the operator norm

$$\|A^pB^p\| \leq \|AB\|^p \qquad \text{for all } 0 < p < 1 \tag{1}$$

is equivalent to the Löwner-Heinz inequality (cf.[16])

$$A \geq B \geq 0 \quad \text{implies} \quad A^p \geq B^p \quad \text{for all } 0 < p < 1 \tag{2}$$

(cf. [7]). In [1], Araki showed a trace inequality which entailed the following inequality:

$$\|B^pA^pB^p\| \leq \|BAB\|^p \quad \text{for all } 0 < p < 1. \tag{3}$$

Moreover, it was shown in [8, 2] that the Cordes inequality (1) is equivalent to Araki's inequality (3).

On the other hand, Furuta [11] showed the following Kantorovich type inequalities of the Löwner-Heinz inequality (2): If A and B are positive operators such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$A \geq B \geq 0 \quad \text{implies} \quad K(m,M,p)A^p \geq B^p \quad \text{for all } p > 1, \tag{4}$$

where a generalized Kantorovich constant $K(m,M,p)$ [5, 9, 13] is defined as

(5)
$$K(m,M,p) = \frac{mM^p - Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p-m^p}{mM^p-Mm^p}\right)^p \qquad \text{for all real numbers } p.$$

We here cite Furuta's textbook [12] as a pertinent reference to Kantorovich inequalities.

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