On the Number of Places of Function Fields and Congruence Zeta Functions

Takehiro HASEGAWA

Abstract

In the first half, we determine completely the numbers of any prime degree places of maximal function fields in the Hasse-Weil sense, and we present a new characterization of the Hermitian function field. In the latter half, in fact we count such numbers for three algebraic function fields. The first and the second examples are generalizations of the Hermitian function field, which are maximal. The last example is the Klein Quartic function field over the field of two elements. This is not maximal.

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1 Introduction

Let q be a power of a prime number and $\mathbb{F}_q$ the finite field of size q. We consider an algebraic function field F of one variable with full constant field $\mathbb{F}_q$. Let $N(F)$ be the number of rational places (i.e., places of degree one) of $F/\mathbb{F}_q$, and let $B_n = B_n(F)$ be the number of places of degree n of $F/\mathbb{F}_q$. Of course $N(F)$ is equal to $B_1(F)$. Let $g = g(F)$ be the genus of an algebraic function field F. It is well-known (for example, [4] Hasse-Weil Bound) that

$$|N(F) - (q + 1)| \leq 2g(F) \cdot \sqrt{q}.$$

One is often interested in algebraic function fields with many rational places. So we introduce the following notion.

Definition. An algebraic function field $F/\mathbb{F}_q$ is called maximal (resp. minimal) if

$$N(F) = q + 1 + 2g(F) \cdot \sqrt{q} \quad (\text{resp.} \quad N(F) = q + 1 - 2g(F) \cdot \sqrt{q}).$$

In this paper, we show the following theorem and corollary.

Theorem. Assume that F is a maximal function field over $\mathbb{F}_{q^2}$ of genus g. Then,

(1)

$$B_2(F) = \frac{q(q + 1)(q^2 - q - 2g)}{2}$$

(2)

$$B_l(F) = \frac{q(q^{l-1} - 1)(q^l + q + 2g)}{l} \quad \text{for every odd prime number } l.$$

By the theorem, it is clear that the number $B_l(F)$ is not zero for any maximal function field F and any odd prime number l. We have the following corollary for $l = 2$.