Nihonkai Math. J.
Vol.14(2003), 31-42

Extremals for families of plane quasiconformal mappings

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Abstract

Let $\mathscr{F}(K)$ be the family of K-quasiconformal mappings from the Riemann sphere $\mathbb{C}^{\#}$ onto $\mathbb{C}^{\#}$, which preserve reals, and moreover, which have three fixed points, $-1, 0$, and ∞. For real t let $\lambda(K,t)$ and $\nu(K,t)$ be the supremum and the infimum, respectively, of the values $f(t)$ for f ranging over the family $\mathscr{F}(K)$. Among others we shall express $X(K,t)$ for $X = \lambda, \nu$, in terms of extremals for various families of K-quasiconformal self-mappings of $\mathbb{C}^{\#}$.

1 Introduction

Let $\mathscr{Q} = \mathscr{Q}(K)$ be the family of all the K-quasiconformal mappings from the Riemann sphere $\mathbb{C}^{\#} = \{|z| \leqslant +\infty\}$ onto $\mathbb{C}^{\#}$. Three families with the inclusion formulae $\mathscr{F} \subset \mathscr{G} \subset \mathscr{H}$ are then defined by $\mathscr{H} = \mathscr{H}(K) = \{ f \in \mathscr{Q};\ f(0) = 0, f(\infty) = \infty\}$; $\mathscr{G} = \mathscr{G}(K) = \{ f \in \mathscr{H};\ f(-1) = -1\}$; $\mathscr{F} = \mathscr{F}(K) = \{ f \in \mathscr{G};\ f(\mathbb{R}) = \mathbb{R}\}$, where $\mathbb{R}$ is the set of all the real numbers, so that $\mathbb{C} = \mathbb{R}^2$ is the complex plane.

In [KY] we studied

$$\lambda(K,t) = \sup_{f \in \mathscr{F}(K)} f(t) \qquad \text{and} \qquad \nu(K,t) = \inf_{f \in \mathscr{F}(K)} f(t)$$

for $t \in \mathbb{R}$ in detail. Since $\mathscr{F}(K)$ is normal, $\lambda(K,t)$ and $\nu(K,t)$ are the maximum and the minimum, respectively. In particular, $\nu(K,t) \leqslant t \leqslant \lambda(K,t)$ for all $t \in \mathbb{R}$ and trivially, $X(K,t) = t$ for $X = \lambda, \nu$, and $t = -1, 0, \infty$; moreover, $X(1,t) \equiv t$. Furthermore, $\nu(K,t) > 0$ for all $t > 0$. For a fixed $K \geqslant 1$ the function $X(K,t)$ is increasing for $t \in \mathbb{R}$. For fixed $t > 0$ the functions $\lambda(K,t)$ and $\nu(K,t)$ are increasing and decreasing functions of $K \geqslant 1$,

2000 *Mathematics Subject Classification*: Primary 30C62; Secondary 30C75.
Key Words and Phrases. The family $\mathscr{Q}$ of quasiconformal self-mappings of the Riemann sphere; subfamilies $\mathscr{F}, \mathscr{G}$, and $\mathscr{H}$ of $\mathscr{Q}$; maximum and minimum of $f(t)$ for varying $f \in \mathscr{F}$.