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Transcendental entire solution of some q-difference equation

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Abstract

We treat linear q-difference equations with polynomial coefficients, in which $q=e^{2\pi\lambda i}, \lambda\in(0,1)\setminus\mathbb{Q}$. Supposing that there is a transcedental entire solution $f(z)$ for this equation, we will show that $f(z)$ takes any finite value infinitely often in any sector.

Keywords and phrases: linear q-difference equation, entire function, rationality and irrationality.

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1 Introduction

We consider here a q-difference equation

(1.1) $$b_p(z)f(q^pz)+\cdots+b_0(z)f(z)=\mathfrak{b}(z),\quad b_j(z),\ \mathfrak{b}(z)\in\mathbb{C}[z],$$

with $b_j(z)=\sum_{k=0}^{B_j}b_k^{(j)}z^k\ \ (b_{B_j}^{(j)}\neq 0), 0\leq j\leq p$, in which we suppose that $|q|=1$, i.e., $q=e^{2\pi i\lambda}$. Further we suppose that

(1.2) $$q=e^{2\pi i\lambda},\quad \lambda\in(0,1)\setminus\mathbb{Q}.$$

The equation (1.1) with q in (1.2) may have transcendental entire solution. In fact, Driver et al. [2] p. 474 showed that there exists a pair (q,A), q in (1.2) and $|A|=1$, such that the equation

(1.3) $$qzf(qz)+(1-Az)f(z)=1$$

has a transcendental entire solution $f(z)$. See also [6].

By the way, Ramis [7] questioned whether (1.1) with q in (1.2) would have transcendental entire solution which also satisfies a linear differential equation.

Here we will consider some properties of solutions of (1.1) with q in (1.2).

First, we introduce some notations: Put $B^*=\max_{0\leq j\leq p}B_j$ ($B_j=\deg[b_j(z)]$) and $j_1<\cdots<j_\tau$ be such that $B^*=B_{j_t}$ ($1\leq t\leq\tau$) with $B_j<B^*$ ($j\neq j_t$). Write $b_t=b_{B^*}^{(j_t)}$ and set

(1.4) $$\phi(z)=\sum_{t=1}^{\tau}b_tz^{j_t-j_1}=0.$$