Nihonkai Math. J.
Vol.13(2002), 57-66

UNIT TANGENT BUNDLE OVER TWO-DIMENSIONAL REAL PROJECTIVE SPACE

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ABSTRACT. In this paper, we prove that the unit tangent bundle over the two-dimensional real projective space is isometric to a lens space. Also, we characterize the Killing vector fields and the geodesics of this bundle in terms of the geometry of the base space.

1. INTRODUCTION

It has been proven by Klingenberg and Sasaki [1] that the unit tangent bundle over a unit two-sphere is isometric to the three-dimensional real projective space of constant curvature 1/4. The authors also studied the geodesics of the bundle. In this paper, we prove that the unit tangent bundle over two-dimensional real projective space is isometric to a lens space. Then we describe the Killing vector fields on this bundle in terms of the geometry of the base space, and characterize the geodesics of the bundle using the Killing vector fields on the base space.

Let $S^n(k)$ denote an n-dimensional sphere of radius $1/\sqrt{k}$ in the Euclidean $(n+1)$-space. We assume that the center of $S^n(k)$ is fixed at the origin of the Euclidean space, and that $S^n(k)$ is endowed with the standard metric. The sectional curvatures of $S^n(k)$ are constant, k. We then denote by $\boldsymbol{RP}^n(k)$ the real projective space that is given by identifying the antipodal points of $S^n(k)$. In order to define lens spaces, we identify the point (x^1, x^2, x^3, x^4) of $\boldsymbol{R}^4$ with the point $(x^1+\sqrt{-1}x^2, x^3+\sqrt{-1}x^4)$ of $\boldsymbol{C}^2$. For any relatively prime integers p, q satisfying $1 \leq q < p$, an isometry of $S^3(k)$ is given by

$$(Z^1, Z^2) \longmapsto (Z^1 e^{2\pi\sqrt{-1}/p}, Z^2 e^{2\pi\sqrt{-1}q/p}), \quad \text{where } (Z^1, Z^2) \in S^3(k) \subset \boldsymbol{C}^2.$$

Let $\Gamma(p,q)$ denote the transformation group generated by this isometry. The quotient space $S^3(k)/\Gamma(p,q)$ is then called the lens space of type (p,q).

For a Riemannian manifold (M, g), let $U(M)$ denote the unit tangent bundle over M, whose total space is the tangent vectors of length one. We assume the Sasaki metric on $U(M)$. Let $\mathfrak{i}(M)$ denote the Lie algebra of the Killing vector fields on M. Then we have

1991 *Mathematics Subject Classification.* Primary 53C25; Secondary 53C22;.
Key words and phrases. unit tangent bundle, lens space, Killing vector field, geodesic.