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Group von Neumann algebras associated with locally compact groups

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Abstract

The relation between two semi-finite von Neumann algebras associated with a locally compact group is investigated.

1 Introduction

Let G be a locally compact group. Then it is known that there exists an action α of G on the additive group $\mathbf{R}$ of real numbers such that the semi-direct product $\mathbf{R} \times_\alpha G$ is unimodular. Hence the group von Neumann algebra $\lambda(\mathbf{R} \times_\alpha G)$ generated by the left regular representation of $\mathbf{R} \times_\alpha G$ is semi-finite. On the other hand, using the modular action σ on the group von Neumann algebra $\lambda(G)$ generated by the left regular representation of G, we also have a semi-finite von Neumann algebra $\lambda(G) \times_\sigma \mathbf{R}$, the crossed product of $\lambda(G)$ by σ.

In this note, we shall compare the structures of these two semi-finite von Neumann algebras.

2 Results

We first recall some basic facts on locally compact groups and von Neumann algebras (see standard text-books in the reference).

Let G be a locally compact group with a fixed left invariant Haar measure and let Δ be the modular function of G. If we define an action α on the additive group $\mathbf{R}$ of real numbers by

$$\alpha_g(t) = \Delta(g)\, t,\ t \in \mathbf{R},\ g \in G,$$

then the semi-direct product $\mathbf{R} \times_\alpha G$ is unimodular, so that the group von Neumann algebra $\lambda(\mathbf{R} \times_\alpha G)$ generated by the left regular representation of $\mathbf{R} \times_\alpha G$ is semi-finite.

On the other hand, let $\lambda(G)$ be the group von Neumann algebra generated by the left regular representation of G and take a natural weight φ on $\lambda(G)$ such that the modular operator Δ_φ arising from φ is given by

$$(\Delta_\varphi^{it}\xi)(g) = \Delta(g)^{it}\xi(g),\ \xi \in L^2(G),\ g \in G,\ t \in \mathbf{R}.$$

Then the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbf{R}}$ on $\lambda(G)$ is computed by

$$\sigma_t^\varphi(\lambda_g) = \Delta(g)^{it}\lambda_g,\ g \in G,\ t \in \mathbf{R},$$